

Refined forms of Oppenheim and Cusa–Huygens type inequalities

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ABSTRACT. We refine Oppenheim’s inequality as well as generalized Cusa–Huygens type inequalities established recently by some researchers. One of the results where the bounds of $\sin x/x$ are tractable will be used to obtain a sharp version of Yang’s inequality.

1. Introduction

The famous Cusa–Huygens inequality [7, 12, 16–18] is known as

$$\frac{\sin x}{x} < \frac{2 + \cos x}{3}, \quad x \in (0, \pi/2).$$

Its hyperbolic version, sometimes called hyperbolic Cusa–Huygens inequality [12], is stated as follows:

$$\frac{\sinh x}{x} < \frac{2 + \cosh x}{3}, \quad x \neq 0. \quad (1.1)$$

Obtaining extended and generalized sharp versions of the above inequalities has been the interest among many researchers in recent years. In [5, 16] the following two sided inequality has been obtained:

$$\left(\frac{2 + \cos x}{3}\right)^a < \frac{\sin x}{x} < \frac{2 + \cos x}{3}, \quad x \in (0, \pi/2), \quad (1.2)$$

with the best positive constants $a \approx 1.11374$ and 1.

Sándor and Oláh-Gál [18, Theorems 1 and 2] proved the inequalities

$$\frac{2 + \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + (\pi/2) \cos x}{\pi}, \quad x \in (0, \pi/2). \quad (1.3)$$

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For $x \in (-\pi/2, \pi/2)$, the double inequality

$$\frac{p-1+\cos x}{p} < \frac{\sin x}{x} < \frac{2+\cos x}{3}, \quad (1.4)$$

where $p \approx 2.75194$, is due to Bhayo et al. [3]. Left inequality in (1.4) is sharper than the corresponding inequalities in (1.2) and (1.3). For $x \in (0, \pi/2)$, the two sided inequality

$$\frac{p-1+\cos x}{p} < \frac{\sin x}{x} < \frac{2+(4/\pi)\cos x}{\pi}, \quad (1.5)$$

where $p \approx 2.75194$, refines (1.3). It gives an optimal answer to the Oppenheim's problem [13]. It is already discussed in [4] and proved by Qi et al. [15] in a more general case.

On the other side, as a natural approach, Mortici [11] established trigonometric-polynomial bounds for $\sin x/x$:

$$\frac{2+\cos x}{3} - \frac{x^4}{180} < \frac{\sin x}{x} < \frac{2+\cos x}{3} - \frac{x^4}{180} + \frac{x^6}{3780}, \quad x \in (0, \pi/2). \quad (1.6)$$

Recently, Malešević et al. [9] extended and generalized (1.6) as follows. If $x \in (0, \pi/2)$ and n is a natural number, then

$$\begin{aligned} \frac{2+\cos x}{3} + \sum_{k=2}^{2n} (-1)^{k+1} B(k) x^{2k} &< \frac{\sin x}{x} \\ &< \frac{2+\cos x}{3} + \sum_{k=2}^{2n+1} (-1)^{k+1} B(k) x^{2k}, \end{aligned} \quad (1.7)$$

where $B(k) = 2(k-1)/[3(2k+1)!]$.

Putting $n = 1$ in (1.7), the authors of the paper [9] obtain (1.6). Putting $n = 2$, they obtain for $x \in (0, \pi/2)$ the inequalities

$$\begin{aligned} \frac{2+\cos x}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} &< \frac{\sin x}{x} \\ &< \frac{2+\cos x}{3} - \frac{x^4}{180} + \frac{x^6}{3780} - \frac{x^8}{181440} + \frac{x^{10}}{14968800}. \end{aligned} \quad (1.8)$$

In this paper, we give simpler alternative proofs of (1.4) and (1.5) as well as establish new extended refined forms of the inequalities listed above. We also aim to improve Yang's inequality [19] in the last section.

2. Preliminaries and lemmas

We start by recalling the power series expansions

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad x \in \mathbb{R}, \quad (2.1)$$

and

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \quad x \in \mathbb{R}. \tag{2.2}$$

The following lemmas will be applied to prove the main results of this paper.

Lemma 1 (The l'Hôpital monotonicity rule [1]). *Let $f_1, f_2 : [p, q] \rightarrow \mathbb{R}$ be two continuous functions which are derivable on (p, q) and $f_2'(x) \neq 0$ for any $x \in (p, q)$. Let*

$$A(x) = \frac{f_1(x) - f_1(p)}{f_2(x) - f_2(p)} \text{ and } B(x) = \frac{f_1(x) - f_1(q)}{f_2(x) - f_2(q)}.$$

If f_1'/f_2' is increasing (or decreasing) on (p, q) , then the functions $A(x)$ and $B(x)$ are also increasing (or decreasing) on (p, q) . The strictness of the monotonicity in the conclusion depends on the strictness of the monotonicity of f'/g' .

From Lemma 1 we immediately get the following result.

Lemma 2. *For $q > 0$ and $m \geq 1$ let $f_1, f_2 : (0, q) \rightarrow \mathbb{R}$ be continuous and m -derivable functions such that $f_i^{(k)}$ ($i = 1, 2; k = 1, \dots, m - 1$) are continuous. Suppose that $f_2^{(k)}(x) \neq 0$ for $k = 1, \dots, m$ and $x \in (0, q)$. If $f_i(0) = f_i'(0) = \dots = f_i^{(m-1)}(0) = 0$ ($i = 1, 2$) and $f_1^{(m)}/f_2^{(m)}$ is increasing (or decreasing) on $(0, q)$, then the function $f_1(x)/f_2(x)$ is also increasing (or decreasing) on $(0, q)$.*

Lemma 3 (see [2], Lemma 2). *The function $H(x) = \frac{\sin x - x \cos x}{x^2 \sin x}$ is strictly positive increasing in $(0, \pi/2)$.*

Lemma 4. *The function $\phi(x) = \frac{\sinh x - x \cosh x}{x^2 \sinh x}$ is strictly negative increasing for $x > 0$.*

Proof. The function $\phi(x)$ is clearly negative as $\sinh x/x < \cosh x$ for $x > 0$ (see [10]). Let us now set

$$\phi(x) = \frac{\sinh x - x \cosh x}{x^2 \sinh x} = \frac{\phi_1(x)}{\phi_2(x)}.$$

Here the functions $\phi_1(x) = \sinh x - x \cosh x$ and $\phi_2(x) = x^2 \sinh x$ satisfy the conditions of Lemma 2 with $m = 2$, and

$$\frac{\phi_1''(x)}{\phi_2''(x)} = \frac{-\cosh x}{x \sinh x + 3 \cosh x} = \frac{-1}{x \tanh x + 3}.$$

It is well known that $x \tanh x$ is increasing, implying that $x \tanh x + 3$ is increasing. By Lemma 2, $\phi(x)$ is a strictly increasing function for $x > 0$. \square

3. Main results and proofs

We first give a simpler alternative proof of (1.4).

Theorem 1 (see [3], Theorem 1). *If $x \in (-\pi/2, \pi/2)$, then*

$$\frac{2 + (\pi - 2) \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}.$$

Proof. It is enough to prove the result for $x \in (0, \pi/2)$. Let us set

$$f(x) = \frac{\frac{\sin x}{x} - \cos x}{\cos x - 1} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \sin x/x - \cos x$ and $f_2(x) = \cos x - 1$ satisfy $f_1(0+) = 0$ and $f_2(0) = 0$. Since

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\frac{x \cos x - \sin x}{x^2} + \sin x}{-\sin x} = \frac{\sin x - x \cos x}{x^2 \sin x} - 1,$$

by Lemmas 1 and 3, $f(x)$ is strictly increasing in $(0, \pi/2)$. We end the proof with $f(0+) = -2/3$ and $f(\pi/2) = -2/\pi$. \square

In the next result, we extend the inequality (1.1).

Theorem 2. *If $x \in (0, \lambda)$ with $\lambda > 0$, then*

$$-\delta + (1 + \delta) \cosh x < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3},$$

where $\delta = (\sinh \lambda/\lambda - \cosh \lambda)/(\cosh \lambda - 1)$.

Proof. Let us consider

$$f(x) = \frac{\frac{\sinh x}{x} - \cosh x}{\cosh x - 1} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \sinh x/x - \cosh x$ and $f_2(x) = \cosh x - 1$ with $f_1(0+) = 0$ and $f_2(0) = 0$. We have

$$\frac{f_1'(x)}{f_2'(x)} = \frac{x \cosh x - \sinh x}{x^2 \sinh x} - 1.$$

By Lemmas 1 and 4, $f(x)$ is strictly decreasing in $(0, \lambda)$. The desired result follows from $f(\lambda) = \delta$ and $f(0+) = -2/3$. \square

In the following theorem we present another simple proof of (1.5).

Theorem 3 (se [4, 15]). *If $x \in (0, \pi/2)$, then*

$$\frac{2 + (\pi - 2) \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + (4/\pi) \cos x}{\pi}.$$

Proof. Let us set

$$f(x) = \frac{\frac{\sin x}{x} - \frac{2}{\pi}}{\cos x} = \frac{f_1(x)}{f_2(x)},$$

where $f_1(x) = \sin x/x - 2/\pi$ and $f_2(x) = \cos x$ with $f_1(\pi/2) = 0$ and $f_2(\pi/2) = 0$. The function

$$\frac{f_1'(x)}{f_2'(x)} = \frac{\sin x - x \cos x}{x^2 \sin x}$$

is strictly increasing in $(0, \pi/2)$ by Lemma 3. So $f(x)$ is also strictly increasing in $(0, \pi/2)$ by Lemma 1. Therefore, the proof of Theorem 3 follows with $f(0+) = (\pi - 2)/\pi$ and $f(\pi/2-) = 4/\pi^2$ by l'Hôpital's rule. \square

Motivated by (1.6), we now establish trigonometric-polynomial bounds for $\sin x/x$ with a different approach.

Theorem 4. *If $x \in (0, \pi/2)$, then*

$$\frac{2 + \cos x}{3} - \frac{x^4}{a} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} - \frac{x^4}{b} \tag{3.1}$$

with the constants $a = 180$ and $b \approx 202.618886$.

Proof. Let us set

$$f(x) = \frac{-3x^5}{3 \sin x - 2x - x \cos x} = \frac{f_1(x)}{f_2(x)}.$$

It is not difficult to see that the functions $f_1(x) = -3x^5$ and $f_2(x) = 3 \sin x - 2x - x \cos x$ satisfy the conditions of Lemma 2 with $q = \pi/2$ and $m = 3$. Here we have

$$\frac{f_1'''(x)}{f_2'''(x)} = 180 \frac{x}{\sin x}$$

which is clearly increasing in $(0, \pi/2)$. Thus by Lemma 2, the function $f(x) = f_1(x)/f_2(x)$ is also increasing in $(0, \pi/2)$. Since $f(0+) = 180$ and $f(\pi/2) \approx 202.618886$, we get (3.1). \square

The hyperbolic version of (3.1) is given in next theorem.

Theorem 5. *If $x \in (0, \lambda)$, where $\lambda > 0$, then*

$$\frac{2 + \cosh x}{3} - \frac{x^4}{m} < \frac{\sinh x}{x} < \frac{2 + \cosh x}{3} - \frac{x^4}{n}$$

with the constants $m = -3\lambda^5/(3 \sinh \lambda - 2\lambda - \lambda \cosh \lambda)$ and $n = 180$.

Proof. The proof is omitted since it is similar to the one of Theorem 4. \square

A refinement of upper bounds of (1.3) and (1.5) can be seen below.

Theorem 6. For $x \in (0, \pi/2)$, one has

$$\frac{\sin x}{x} < \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{\pi - 3}{3\pi} x^2. \quad (3.2)$$

Proof. As in the proofs of previous theorems, let us set

$$f(x) = \frac{\pi \sin x - 2x - (\pi - 2)x \cos x}{\pi x^3} = \frac{f_1(x)}{f_2(x)}.$$

Here $f_1(x) = \pi \sin x - 2x - (\pi - 2)x \cos x$ and $f_2(x) = \pi x^3$ satisfy the conditions of Lemma 2 with $q = \pi/2$ and $m = 3$. We have

$$\frac{f_1'''(x)}{f_2'''(x)} = \frac{-\pi \cos x + (\pi - 2)(\cos x - x \sin x)}{6\pi} = \frac{f_3(x)}{6\pi},$$

where

$$f_3(x) = -\pi \cos x + (\pi - 2)(3 \cos x - x \sin x).$$

The derivative

$$f_3'(x) = (-3\pi + 8) \sin x - (\pi - 2)x \cos x < 0$$

implies that $f_3(x)$ is strictly decreasing in $(0, \pi/2)$. Therefore, by Lemma 2 $f(x)$ is also decreasing and hence

$$f(0+) > \frac{\pi \sin x - 2x - (\pi - 2)x \cos x}{\pi x^3}.$$

With $f(0+) = (\pi - 3)/(3\pi)$, the result is proved. \square

Remark 1. There is no strict comparison between the corresponding bounds of (1.5) and (3.2). However when x is close to zero, (3.2) is better since the inequality becomes an equality for $x = 0$.

Combining (1.5) and (3.2) we have, for $x \in (0, \pi/2)$, that

$$\frac{2 + (\pi - 2) \cos x}{\pi} < \frac{\sin x}{x} < \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{\pi - 3}{3\pi} x^2. \quad (3.3)$$

In the following theorem, we extend and refine the bounds of (3.3).

Theorem 7. If $n \in \mathbb{N}$ (the set of natural numbers) and $x \in (0, \pi)$, then we have

$$M(x) < \frac{\sin x}{x} < N(x) \quad (3.4)$$

where

$$M(x) = \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{(k\pi - 2k - 1)}{(2k + 1)!} x^{2k}$$

and

$$N(x) = \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{(k\pi - 2k - 1)}{(2k + 1)!} x^{2k}.$$

Proof. Let us set

$$f(x) = \frac{\sin x}{x} - M(x).$$

Utilizing (2.1) and (2.2), we get

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k+1)!} - \frac{2}{\pi} - \frac{\pi-2}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \\ &\quad - \frac{2}{\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} \\ &= \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} - \frac{2}{\pi} - \frac{2}{\pi} \sum_{k=1}^{2n} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} \\ &= -\frac{2}{\pi} R, \end{aligned}$$

where

$$R = \sum_{k=2n+1}^{\infty} (-1)^k \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k}.$$

Hence R can be viewed as a rest of the alternating series S given by

$$S = \sum_{k=2}^{\infty} (-1)^k a_k,$$

with $a_k = [(k\pi - 2k - 1)/(2k + 1)!]x^{2k} > 0$. Now for $k \geq 2$ and $x \in (0, \pi)$ we have

$$(k\pi - 2k - 1)(4k^2 + 10k + 6 - x^2) \geq (2\pi - 5)(42 - \pi^2) > (\pi - 2)x^2.$$

This gives

$$\frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} > \frac{(k+1)\pi - 2(k+1) - 1}{(2k+3)!} x^{2(k+1)}.$$

Hence $|a_k| > |a_{k+1}|$. Moreover, for $x \in (0, \pi)$ we have

$$\lim_{k \rightarrow \infty} |a_k| = \lim_{k \rightarrow \infty} \frac{k\pi - 2k - 1}{(2k+1)!} x^{2k} = \frac{\pi - 2}{2} \lim_{k \rightarrow \infty} \frac{x^{2k}}{(2k)!} = 0.$$

This implies that S is convergent and, by a special result on alternating series, R has the same sign to the first term of its sum, i.e., $(-1)^{2n+1} a_{2n+1} < 0$. So $R < 0$, implying that $f(x) > 0$, and

$$M(x) < \frac{\sin x}{x}. \tag{3.5}$$

Similarly, set

$$\begin{aligned} g(x) &= N(x) - \frac{\sin x}{x} \\ &= \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \sum_{k=1}^{2n+1} (-1)^{k+1} \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k} - \frac{\sin x}{x}. \end{aligned}$$

Using again (2.1) and (2.2) and proceeding as in the case of $f(x)$, we obtain

$$g(x) = \frac{2}{\pi} R_*,$$

where

$$R_* = \sum_{k=2n+2}^{\infty} (-1)^k \frac{k\pi - 2k - 1}{(2k + 1)!} x^{2k}.$$

Let us observe that R_* is a rest of the alternating series S . By applying similar arguments to R , we get $R_* > 0$, implying that $g(x) > 0$ and hence

$$\frac{\sin x}{x} < N(x). \quad (3.6)$$

The proof follows from (3.5) and (3.6). \square

Some particular cases of Theorem 7 are presented below. If $x \in (0, \pi)$, then by putting $n = 1$ in (3.4) we get

$$\begin{aligned} &\frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \left[\frac{\pi - 3}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 \right] < \frac{\sin x}{x} \\ &< \frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \left[\frac{\pi - 3}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 + \frac{3\pi - 7}{7!} x^6 \right]. \end{aligned}$$

By putting $n = 2$ in (3.4) we get

$$\begin{aligned} &\frac{2 + (\pi - 2) \cos x}{\pi} + \frac{2}{\pi} \left[\frac{\pi - 3}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 + \frac{3\pi - 7}{7!} x^6 - \frac{4\pi - 9}{9!} x^8 \right] \\ &< \frac{\sin x}{x} < \frac{2 + (\pi - 2) \cos x}{\pi} \\ &+ \frac{2}{\pi} \left[\frac{(\pi - 3)}{3!} x^2 - \frac{2\pi - 5}{5!} x^4 + \frac{3\pi - 7}{7!} x^6 - \frac{4\pi - 9}{9!} x^8 + \frac{5\pi - 11}{11!} x^{10} \right]. \end{aligned}$$

Next we claim that, the generalized extended bounds for $\sin x/x$ in the following theorem are sharper than the corresponding bounds in (1.7).

Theorem 8. *If $m = 2n - 1$, where $n \in \mathbb{N}$ and $x \in (0, \pi)$, then we have*

$$F(x) < \frac{\sin x}{x} < G(x), \quad (3.7)$$

where

$$F(x) = \frac{2m + \cos x}{2m + 1} + \frac{2}{2m + 1} \sum_{k=1}^{m+1} \frac{k - m}{(2k + 1)!} (-1)^{k+1} x^{2k}$$

and

$$G(x) = \frac{(2m + 2) + \cos x}{2m + 3} + \frac{2}{2m + 3} \sum_{k=1}^{m+2} \frac{k - m - 1}{(2k + 1)!} (-1)^{k+1} x^{2k}.$$

Proof. Let us set

$$\begin{aligned} f(x) &= \frac{\sin x}{x} - F(x) \\ &= \frac{\sin x}{x} - \frac{2m}{2m + 1} - \frac{1}{2m + 1} \cos x - \frac{2}{2m + 1} \sum_{k=1}^{m+1} \frac{k - m}{(2k + 1)!} (-1)^{k+1} x^{2k}. \end{aligned}$$

Utilizing (2.1) and (2.2), after some calculations, we get

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k} - \frac{2m}{2m + 1} - \frac{1}{2m + 1} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\ &\quad - \frac{2}{2m + 1} \sum_{k=1}^{m+1} \frac{k - m}{(2k + 1)!} (-1)^{k+1} x^{2k} \\ &= \frac{2}{2m + 1} \sum_{k=m+2}^{\infty} \frac{k - m}{(2k + 1)!} (-1)^{k+1} x^{2k} = \frac{2}{2m + 1} x^{2m} T, \end{aligned}$$

where

$$T = \sum_{k=2}^{\infty} \frac{k}{(2k + 2m + 1)!} (-1)^k x^{2k}.$$

Then T can be viewed as a rest of the alternating series

$$U = \sum_{k=1}^{\infty} (-1)^k b_k,$$

with $b_k = (k/(2k + 2m + 1)!)x^{2k}$. Now, for $k \geq 1$ and $x \in (0, \pi)$, we have

$$k[4(k + m)^2 + 10(k + m) + 6] - kx^2 \geq k(42 - \pi^2) > x^2.$$

This gives

$$\frac{k}{(2k + 2m + 1)!} x^{2k} > \frac{k + 1}{(2k + 2m + 3)!} x^{2k+2}.$$

Hence $|b_k| > |b_{k+1}|$. Moreover, for $x \in (0, \pi)$, we have

$$\lim_{k \rightarrow \infty} |b_k| = \lim_{k \rightarrow \infty} \frac{k}{(2k + 2m + 1)!} x^{2k} = \frac{1}{2} \lim_{k \rightarrow \infty} \frac{x^{2k}}{(2k + 2m)!} = 0.$$

This implies that U is convergent and $(-1)^2 b_2 > 0$, so $T > 0$. Thus, $f(x) > 0$ and

$$F(x) < \frac{\sin x}{x}. \quad (3.8)$$

Similarly, by setting $g(x) = G(x) - \frac{\sin x}{x}$, proceeding as in the case of $f(x)$ and applying the same arguments, we get

$$\frac{\sin x}{x} < G(x). \quad (3.9)$$

The proof follows from (3.8) and (3.9). \square

Some particular cases of Theorem 8 are presented below. If $x \in (0, \pi)$, then by putting $n = 1$ (so $m = 1$) in (3.7) we obtain

$$\frac{2 + \cos x}{3} - \frac{x^4}{180} < \frac{\sin x}{x} < \frac{4 + \cos x}{5} - \frac{x^2}{15} + \frac{x^6}{12600}. \quad (3.10)$$

By putting $n = 2$ (so $m = 3$) in (3.7) we get

$$\begin{aligned} \frac{6 + \cos x}{7} - \frac{2x^2}{21} + \frac{x^4}{420} - \frac{x^8}{1270080} &< \frac{\sin x}{x} \\ &< \frac{8 + \cos x}{9} - \frac{x^2}{9} + \frac{x^4}{270} - \frac{x^6}{22680} + \frac{x^{10}}{179625600}. \end{aligned} \quad (3.11)$$

For comparison between particular cases, it can be verified by any mathematical software that the bounds of $\sin x/x$ in (3.10) and (3.11) are sharper than the corresponding bounds in (1.6) and (1.8), respectively. Moreover, all the bounds are trigonometric-polynomial. Therefore, considering the number of terms and the degrees of polynomials in respective bounds, we conclude that our bounds are better.

4. An application

Well-known Yang's inequality [19] can be stated as follows. If $A_1 > 0$, $A_2 > 0$, $A_1 + A_2 \leq \pi$, and $0 \leq \lambda \leq 1$, then

$$\cos^2 \lambda A_1 + \cos^2 \lambda A_2 - 2 \cos \lambda A_1 \cos \lambda A_2 \cos \lambda \pi \geq \sin^2 \lambda \pi. \quad (4.1)$$

Inequality (4.1) and its generalizations play an important role in the theory of distribution of values. Therefore, many refinements of (4.1) are established so far. For more details we refer the reader to [6, 8, 14, 20] and the references therein.

Recently Chen and Elezović [6] obtained an improved form of Yang's inequality. Let $A_i > 0$ with $\sum_{i=1}^n A_i \leq \pi$, $n \in \mathbb{N}$, $n > 1$, and $0 \leq \mu \leq 1$. Then

$$N(\mu) \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq M(\mu), \quad (4.2)$$

where

$$H_{ij} = \cos^2 \mu A_i + \cos^2 \mu A_j - 2 \cos \mu A_i \cos \mu A_j \cos \mu \pi,$$

$$N(\mu) = \frac{n(n-1)}{2} \pi^2 \left[\frac{1 - (\mu/2)^2}{1 + (\mu/2)^2} \right]^2 \mu^2 \cos^2 \frac{\mu \pi}{2},$$

and

$$M(\mu) = \frac{n(n-1)}{2} \pi^2 \left[\frac{1 - (\mu/2)^3}{1 + 2(\mu/2)^3} \right]^2 \mu^2.$$

Here we refine the Yang’s inequality and hence (4.2) by using the tractable bounds of $\sin x/x$.

Theorem 9. *Let $A_i > 0$ with $\sum_{i=1}^n A_i \leq \pi$, $n \in \mathbb{N}$, $n > 1$, and $0 \leq \mu \leq 1$. Then*

$$P(\mu) \leq \sum_{1 \leq i < j \leq n} H_{ij} \leq Q(\mu), \tag{4.3}$$

where

$$P(\mu) = 2n(n-1) \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right]^2 \mu^2 \cos^2 \frac{\mu \pi}{2}$$

and

$$Q(\mu) = 2n(n-1) \left\{ \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right] + \frac{\mu^2 \pi^2 (\pi-3)}{24} \right\}^2 \mu^2.$$

Proof. We substitute $x = \mu\pi/2$ in (3.3) to get

$$\mu \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right] < \sin \frac{\mu \pi}{2}$$

and

$$\sin \frac{\mu \pi}{2} < \mu \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right] + \frac{\mu^3 \pi^2 (\pi-3)}{24}.$$

Using the inequality (2.13) from [20], for $1 \leq i < j \leq n$ we have

$$4 \sin^2 \frac{\mu \pi}{2} \cos^2 \frac{\mu \pi}{2} \leq H_{ij} \leq 4 \sin^2 \frac{\mu \pi}{2}.$$

We deduce that

$$4 \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right]^2 \mu^2 \cos^2 \frac{\mu \pi}{2} \leq H_{ij} \tag{4.4}$$

$$\leq 4 \left\{ \left[1 + \frac{(\pi-2)}{2} \cos \frac{\mu \pi}{2} \right] + \frac{\mu^2 \pi^2 (\pi-3)}{24} \right\}^2 \mu^2.$$

Summing up all the inequalities in (4.4), we get (4.3). □

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