Simpson’s type inequalities for $\eta$-convex functions via $k$-Riemann–Liouville fractional integrals

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Abstract. We introduce some Simpson’s type integral inequalities via $k$-Riemann–Liouville fractional integrals for functions whose derivatives are $\eta$-convex. These results generalize some results in the literature.

1. Introduction

The well known Simpson’s inequality states as follows.

**Theorem 1.1.** Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times differentiable function on $(a, b)$. Then

$$\left| \int_a^b (t) dt - \frac{b-a}{6} \left[ f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] \right| \leq \frac{(b-a)^5}{2880} \| f^{(4)} \|_\infty,$$

where $\| f^{(4)} \|_\infty = \sup_{x \in (a, b)} | f^{(4)}(x) | < \infty$.

Many authors have studied and provided several generalizations of this inequality over the years. For some results related to the Simpson’s inequality, we refer the interested reader to the papers [1, 2, 5, 9, 10, 15–17].

Gordji et al. [7] introduced the concept of $\eta$-convexity which generalizes the classical concept of convexity.

**Definition 1.2.** A function $f : I \rightarrow \mathbb{R}$ is said to be $\eta$-convex on $I$ with respect to the bifunction $\eta : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ if

$$f(tx + (1-t)y) \leq f(y) + t\eta(f(x), f(y))$$

for all $x, y \in I$ and $t \in [0, 1]$.

**Remark 1.3.** If $\eta(x, y) = x - y$, then we recover the classical notion of convexity.

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For more information on $\eta$-convex functions and some related results, we refer the interested reader to the papers [4, 7, 12] and the references therein.

We complete this section with the definition of the $k$-Riemann–Liouville fractional integrals.

**Definition 1.4** (see [11]). The Riemann–Liouville $k$-fractional integrals of order $\alpha > 0$, for a real-valued continuous function $f$, are defined as

\[ kJ^\alpha_a f(x) = \frac{1}{k \Gamma_k(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) dt, \quad x > a, \]

and

\[ kJ^\alpha_b f(x) = \frac{1}{k \Gamma_k(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) dt, \quad x < b, \]

where $k > 0$, and $\Gamma_k$ is the $k$-gamma function given by

\[ \Gamma_k(x) = \int_0^\infty t^{x-1} e^{-t} t^k dt, \quad \text{Re}(x) > 0 \]

with the properties that $\Gamma_k(x + k) = x\Gamma_k(x)$ and $\Gamma_k(k) = 1$.

**Remark 1.5.** If $k = 1$, then we have the Riemann–Liouville fractional integral of order $\alpha > 0$. If $\alpha = k = 1$, then we get the classical Riemann integral.

For more information and some results related to this integral operator, we refer the interested reader to [8, 11–13] and the references therein.

Our goal in this paper is to provide some Simpson’s type integral inequalities involving the $k$-Riemann–Liouville fractional integral for functions whose first derivatives in absolute value at some powers are $\eta$-convex. Our results generalizes some results in the literature.

2. Main results

To prove our main results, we need the following integral identity via the $k$-Riemann–Liouville fractional integrals.

**Lemma 2.1.** Let $f : [a, b] \to \mathbb{R}$ be a differentiable on $(a, b)$, $a < b$, function. If $f' \in L_1([a, b])$, $n \geq 0$, $\alpha > 0$, and $k > 0$, then the following identity holds:

\[
S(\alpha, n, k) := \frac{1}{6} \left[ f(a) + f(b) + 2f\left(\frac{a + nb}{n + 1}\right) + 2f\left(\frac{na + b}{n + 1}\right) - \Gamma_k(\alpha + k) \left(\frac{n + 1}{b - a}\right) \frac{\alpha}{k} \left[ kJ^\alpha_{\frac{a+nb}{n+1}} f(a) + kJ^\alpha_{\frac{na+b}{n+1}} f(b) \right] - \Gamma_k(\alpha + k) \left(\frac{n + 1}{b - a}\right) \frac{\alpha}{k} \left[ kJ^\alpha_{a} f\left(\frac{na+b}{n+1}\right) + kJ^\alpha_{b} f\left(\frac{a+nb}{n+1}\right) \right] \right]
\]
\[= \frac{b-a}{2(n+1)} \left\{ \int_0^1 \left[ \frac{2(1-t)}{3} - \frac{t^2}{2(1-t)} \right] f' \left( \frac{n+t}{n+1} a + \frac{1-t}{n+1} b \right) dt \right. \\
+ \left. \int_0^1 \left[ \frac{t^2}{3} - \frac{2(1-t)}{3} \right] f' \left( \frac{1-t}{n+1} a + \frac{n+t}{n+1} b \right) dt \right\}.
\]

Proof. Integrating by parts, using change of variables, and the definition of the \(k\)-Riemann–Liouville integral, we have

\[I_1 := \int_0^1 \left[ \frac{2(1-t)}{3} - \frac{t^2}{2(1-t)} \right] f' \left( \frac{n+t}{n+1} a + \frac{1-t}{n+1} b \right) dt = \frac{n+1}{3(b-a)} \left[ f(a) + 2f \left( \frac{na+b}{n+1} \right) \right] - \frac{2\Gamma_k(\alpha+k)(n+1)}{3} \left( \frac{n+1}{b-a} \right)^{\frac{\alpha+1}{\alpha}} \\
\times k J_{\frac{\alpha}{\alpha+1}} \left( \frac{na+b}{n+1} \right) f(a) - \frac{\Gamma_k(\alpha+k)}{3} \left( \frac{n+1}{b-a} \right)^{\frac{\alpha+1}{\alpha}} k J_{\frac{\alpha}{\alpha+1}} \left( \frac{na+b}{n+1} \right)
\]

and

\[I_2 := \int_0^1 \left[ \frac{t^2}{3} - \frac{2(1-t)}{3} \right] f' \left( \frac{1-t}{n+1} a + \frac{n+t}{n+1} b \right) dt = \frac{n+1}{3(b-a)} \left[ f(b) + 2f \left( \frac{a+nb}{n+1} \right) \right] - \frac{2\Gamma_k(\alpha+k)(n+1)}{3} \left( \frac{n+1}{b-a} \right)^{\frac{\alpha+1}{\alpha}} \\
\times k J_{\frac{\alpha}{\alpha+1}} \left( \frac{a+nb}{n+1} \right) f(b) - \frac{\Gamma_k(\alpha+k)}{3} \left( \frac{n+1}{b-a} \right)^{\frac{\alpha+1}{\alpha}} k J_{\frac{\alpha}{\alpha+1}} \left( \frac{a+nb}{n+1} \right) f(b).
\]

Using these equalities, we have that

\[I_1 + I_2 = \frac{2(n+1)}{b-a} S(\alpha, n, k),
\]

which gives the desired identity. \(\square\)

Remark 2.2. If \(k = 1\), then we obtain the identity in Lemma 2.1 of \([15]\).

Theorem 2.3. Under the conditions of Lemma 2.1, suppose that \(|f'|\) is \(\eta\)-convex on \([a, b]\). Then

\[|S(\alpha, n, k)| \leq \frac{b-a}{6(n+1)} \left\{ \mathcal{A}(\alpha, k) (|f'(a)| + |f'(b)|) \\
+ \mathcal{B}(\alpha, n, k) \left( \eta (|f'(a)|, |f'(b)|) + \eta (|f'(b)|, |f'(a)|) \right) \right\},
\]

where

\[Q(\alpha, k) := \frac{2^\frac{1}{\alpha}}{2^\frac{1}{\alpha} + 1}, \quad \mathcal{A}(\alpha, k) := 3 - 4(1 - Q(\alpha, k))^{\frac{\alpha+1}{\alpha}} - 2(\mathcal{Q}(\alpha, k))^{\frac{\alpha+1}{\alpha}},
\]

and 
\[\mathcal{Q}(\alpha, k) := \frac{2^\frac{1}{\alpha}}{2^\frac{1}{\alpha} + 1}, \quad \mathcal{B}(\alpha, n, k) := \frac{2^\frac{1}{\alpha}}{2^\frac{1}{\alpha} + 1} \left( \frac{n+1}{b-a} \right)^{\frac{\alpha+1}{\alpha}}.
\]
and

\[ B(\alpha, n, k) := \frac{3n + 1 - (n + Q(\alpha, k)) \left( 4(1 - Q(\alpha, k))^2 + 2(\alpha, k) \right)}{\frac{4}{\alpha} + 1} + \frac{1 - 4(1 - Q(\alpha, k))^2 + 2(\alpha, k)}{(\frac{4}{\alpha} + 1) (\frac{4}{\alpha} + 2)}. \]

**Proof.** Using Lemma 2.1 and the \( \eta \)-convexity of \( |f'| \), we get

\[
|S(\alpha, n, k)| \leq \frac{b - a}{6(n + 1)} \left\{ \int_0^1 \left| 2(1 - t)^{\frac{2}{\alpha}} - t^{\frac{2}{\alpha}} \right| \left| f' \left( \frac{n + t}{n + 1} a + \frac{1 - t}{n + 1} b \right) \right| dt + \int_0^1 \left| t^{\frac{2}{\alpha}} - 2(1 - t)^{\frac{2}{\alpha}} \right| dt + \frac{1}{n + 1} \eta\left( |f'(a)|, |f'(b)| \right) \right\} 
\]


\[
\int_0^1 \left| 2(1 - t)^{\frac{2}{\alpha}} - t^{\frac{2}{\alpha}} \right| dt = \int_0^{Q(\alpha, k)} \left( 2(1 - t)^{\frac{2}{\alpha}} - t^{\frac{2}{\alpha}} \right) dt + \int_{Q(\alpha, k)}^1 \left( t^{\frac{2}{\alpha}} - 2(1 - t)^{\frac{2}{\alpha}} \right) dt = A(\alpha, k) \tag{2.1}
\]

and

\[
\int_0^1 (n + t) \left| 2(1 - t)^{\frac{2}{\alpha}} - t^{\frac{2}{\alpha}} \right| dt = \int_0^{Q(\alpha, k)} (n + t) \left( 2(1 - t)^{\frac{2}{\alpha}} - t^{\frac{2}{\alpha}} \right) dt + \int_{Q(\alpha, k)}^1 (n + t) \left( t^{\frac{2}{\alpha}} - 2(1 - t)^{\frac{2}{\alpha}} \right) dt = B(\alpha, n, k). \tag{2.2}
\]

\[\square\]
Remark 2.4. If $k = 1$ and $\eta(x, y) = x - y$ in Theorem 2.3, then we recover [15, Theorem 2.2].

Theorem 2.5. Under the conditions of Lemma 2.1, suppose that $|f'|^q$ is $\eta$-convex on $[a, b]$ for $q > 1$. Then we have the inequality

$$
|S(\alpha, n, k)| \leq \frac{b - a}{6(n + 1)} \left( (A(\alpha, k))^{1 - \frac{1}{q}} + \frac{B(\alpha, n, k)}{n + 1} \eta\left(|f'(a)|^q, |f'(b)|^q\right) \right) + (A(\alpha, k))^{\frac{1}{q}}
$$

where $A(\alpha, k)$ and $B(\alpha, n, k)$ are defined in Theorem 2.3.

Proof. Using Lemma 2.1, Hölder’s inequality, and the $\eta$-convexity of $|f'|^q$, we have

$$
|S(\alpha, n, k)|
\leq \frac{b - a}{6(n + 1)} \left\{ \int_0^1 \left| 2(1 - t)^{\frac{q}{2}} - t^{\frac{q}{2}} \right| f'\left(\frac{n + t}{n + 1}a + \frac{1 - t}{n + 1}b\right) dt
\right. \\
+ \left. \int_0^1 \left| t^{\frac{q}{2}} - 2(1 - t)^{\frac{q}{2}} \right| f'\left(\frac{1 - t}{n + 1}a + \frac{n + t}{n + 1}b\right) dt \right\}
\leq \frac{b - a}{6(n + 1)} \left( \int_0^1 \left| t^{\frac{q}{2}} - 2(1 - t)^{\frac{q}{2}} \right| dt \right)^{1 - \frac{1}{q}}
\times \left( \int_0^1 \left| 2(1 - t)^{\frac{q}{2}} - t^{\frac{q}{2}} \right| f'\left(\frac{n + t}{n + 1}a + \frac{1 - t}{n + 1}b\right) dt \right)^{\frac{1}{q}}
\times \left( \int_0^1 \left| t^{\frac{q}{2}} - 2(1 - t)^{\frac{q}{2}} \right| f'\left(\frac{1 - t}{n + 1}a + \frac{n + t}{n + 1}b\right) dt \right)^{\frac{1}{q}}
\leq \frac{b - a}{6(n + 1)} \left( \int_0^1 \left| 2(1 - t)^{\frac{q}{2}} - t^{\frac{q}{2}} \right| dt \right)^{1 - \frac{1}{q}}
\times \left\{ \int_0^1 \left| 2(1 - t)^{\frac{q}{2}} - t^{\frac{q}{2}} \right| \left( |f'(b)|^q + \frac{n + t}{n + 1} \eta\left(|f'(a)|^q, |f'(b)|^q\right) \right) dt \right. \\
+ \left. \int_0^1 \left| t^{\frac{q}{2}} - 2(1 - t)^{\frac{q}{2}} \right| \left( |f'(a)|^q + \frac{n + t}{n + 1} \eta\left(|f'(b)|^q, |f'(a)|^q\right) \right) dt \right\},
$$

which proves, in view of (2.1) and (2.2), the desired inequality. □
Remark 2.6. If \( k = 1 \), and \( \eta(x, y) = x - y \) in Theorem 2.5, then we recover \[15, \text{Theorem 2.4}\].

Theorem 2.7. Under the conditions of Lemma 2.1, suppose that \(|f'|^q\) is \( \eta \)-convex on \([a, b]\) for \( q > 1 \). Then

\[
|S(\alpha, n, k)| \leq \frac{b - a}{6(n + 1)} \left( \frac{2^{\frac{\alpha}{p} + 1}}{\left( \frac{\alpha}{p} + 1 \right) \left( \frac{2^{\frac{\alpha}{p}} + 1}{2^{\frac{\alpha}{p}}} \right)} \right)^{\frac{1}{p}} 
\times \left\{ \left( \left| f'(b) \right|^q + \frac{2n + 1}{2(n + 1)} \eta(|f'(a)|^q, |f'(b)|^q) \right)^{\frac{1}{q}} 
\right. 
\left.+ \left( \left| f'(a) \right|^q + \frac{2n + 1}{2(n + 1)} \eta(|f'(b)|^q, |f'(a)|^q) \right)^{\frac{1}{q}} \right\},
\]

(2.3)

where \( 1/p + 1/q = 1 \) and

\[
C(\alpha, k, p) := \int_0^1 \left| 2(1 - t)\frac{\alpha}{p} - t^\alpha \right|^p dt.
\]

In addition, if \( \alpha/k \in [0, 1] \), then

\[
|S(\alpha, n, k)| \leq \frac{b - a}{6(n + 1)} \left( \frac{2^{\frac{\alpha}{p} + 1}}{\left( \frac{\alpha}{p} + 1 \right) \left( \frac{2^{\frac{\alpha}{p}} + 1}{2^{\frac{\alpha}{p}}} \right)} \right)^{\frac{1}{p}} 
\times \left\{ \left( \left| f'(b) \right|^q + \frac{2n + 1}{2(n + 1)} \eta(|f'(a)|^q, |f'(b)|^q) \right)^{\frac{1}{q}} 
\right. 
\left.+ \left( \left| f'(a) \right|^q + \frac{2n + 1}{2(n + 1)} \eta(|f'(b)|^q, |f'(a)|^q) \right)^{\frac{1}{q}} \right\},
\]

(2.4)

Proof. Using Lemma 2.1, Hölder’s inequality, and the \( \eta \)-convexity of \(|f'|^q|\), we have

\[
|S(\alpha, n, k)| 
\leq \frac{b - a}{6(n + 1)} \left( \int_0^1 \left| 2(1 - t)\frac{\alpha}{p} - t^\alpha \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f' \left( \frac{n + t}{n + 1} a + \frac{1 - t}{n + 1} b \right) \right|^q dt \right)^{\frac{1}{q}} 
\left.+ \left( \int_0^1 \left| 2(1 - t)\frac{\alpha}{p} - t^\alpha \right|^p dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f' \left( \frac{1 - t}{n + 1} a + \frac{n + t}{n + 1} b \right) \right|^q dt \right)^{\frac{1}{q}} \right\}
\leq \frac{b - a}{6(n + 1)} \left( \int_0^1 \left| 2(1 - t)\frac{\alpha}{p} - t^\alpha \right|^p dt \right)^{\frac{1}{p}} 
\times \left\{ \left( \int_0^1 \left| f'(b) \right|^q + \frac{n + t}{n + 1} \eta(|f'(a)|^q, |f'(b)|^q) \right) dt \right\}^{\frac{1}{q}}.
\]
\begin{align*}
&+ \left( \int_0^1 \left( |f'(a)|^q + \frac{n+1}{n+1} \eta \left( |f'(b)|^q, |f'(a)|^q \right) \right) \frac{1}{q} \right) dt \\
&= \frac{b-a}{6(n+1)} \left( \int_0^1 \left( \left| 2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \right|^p dt \right)^{\frac{1}{p}} \right) \\
&\times \left\{ \left( |f'(b)|^q \int_0^1 dt + \frac{1}{n+1} \eta \left( |f'(a)|^q, |f'(b)|^q \right) \int_0^1 (n+t) dt \right)^{\frac{1}{q}} \\
&+ \left( |f'(a)|^q \int_0^1 dt + \frac{1}{n+1} \eta \left( |f'(b)|^q, |f'(a)|^q \right) \int_0^1 (n+t) dt \right)^{\frac{1}{q}} \right\}.
\end{align*}

This gives the inequality (2.3) after simple calculations.

Now, from the assumption \( \alpha/k \in [0, 1] \), we have that

\[ \left| x^{\frac{\alpha}{k}} - y^{\frac{\alpha}{k}} \right| \leq \left| x - y \right|^{\frac{\alpha}{k}} \text{ for all } x, y \in [0, 1]. \]

So, it follows that

\[ \left| 2(1-t)^{\frac{\alpha}{k}} - t^{\frac{\alpha}{k}} \right| = 2 \left( (1-t)^{\frac{\alpha}{k}} - \left( \frac{t}{2^{\frac{k}{p}}} \right)^{\frac{\alpha}{k}} \right) \leq 2 \left| 1 - \left( 1 + \frac{1}{2^{\frac{k}{p}}} \right) t \right|^{\frac{\alpha}{k}}. \]

Thus,

\[ C(\alpha, k, p) \leq \int_0^1 \left( 2 \left| 1 - \left( 1 + \frac{1}{2^{\frac{k}{p}}} \right) t \right|^{\frac{\alpha}{k}} \right)^p dt \]

\[ = 2^p \left( \int_0^{Q(\alpha, k)} \left( 1 - \left( \frac{2^{\frac{k}{p}} + 1}{2^{\frac{k}{p}}} \right) t \right)^{\frac{\alpha}{k}} dt + \int_{Q(\alpha, k)}^1 \left( \left( \frac{2^{\frac{k}{p}} + 1}{2^{\frac{k}{p}}} \right) t - 1 \right)^{\frac{\alpha}{k}} dt \right) \]

\[ = 2^p \left( \frac{2^{\frac{k}{p}}}{\left( \frac{\alpha}{k} p + 1 \right) \left( \frac{2^{\frac{k}{p}}}{2^{\frac{k}{p}}} + 1 \right)} + \frac{2^{\frac{k}{p}}}{\left( \frac{\alpha}{k} p + 1 \right) \left( \frac{2^{\frac{k}{p}}}{2^{\frac{k}{p}}} + 1 \right)} \left( \frac{1}{2^{\frac{k}{p}}} \right)^{\frac{\alpha}{k} p+1} \right) \]

\[ = \frac{2^{\frac{k}{p}}}{\left( \frac{\alpha}{k} p + 1 \right) \left( \frac{2^{\frac{k}{p}}}{2^{\frac{k}{p}}} + 1 \right)}. \]

This proves, in view of (2.3), the inequality (2.4).

\[ \square \]

**Remark 2.8.** If \( k = 1 \), and \( \eta(x, y) = x - y \) in the inequality (2.3) of Theorem 2.7, then we recover [15, Theorem 2.3].

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