Lucas numbers of the form \((\begin{array}{c}2^t \\ k \end{array})\)

Nurettin Irmak and László Szalay

To the memory of Professor József Závoiti

Abstract. Let \(L_m\) denote the \(m\)th Lucas number. We show that the solutions to the diophantine equation \((\begin{array}{c}2^t \\ k \end{array}) = L_m\), in non-negative integers \(t, k \leq 2^t - 1\), and \(m\), are \((t, k, m) = (1, 1, 0), (2, 1, 3),\) and \((a, 0, 1)\) with non-negative integers \(a\).

1. Introduction

As usual, the sequence of Lucas numbers is defined by \(L_0 = 2, L_1 = 1,\) and
\[
L_m = L_{m-1} + L_{m-2}, \quad m \geq 2.
\]
This sequence is known as the associate of Fibonacci sequence.

Now we present a short historical background related to the title problem. The occurrence of figurate numbers in linear recurrences has had a very extensive literature. The first challenging result is due to Cohn [1, 2], and independently to Wyler [18], who proved that the square Fibonacci numbers are \(F_0 = 0, F_1 = F_2 = 1\) and \(F_{12} = 144\). Focusing only on the occurrence of binomial coefficients in binary recurrences, first we mention that Ming [11] proved a conjecture of Hoggatt [5]. Namely, he showed that \(F_0 = 0, F_1 = F_2 = 1, F_4 = 3, F_8 = 21\) and \(F_{10} = 55\) are the only triangular Fibonacci numbers, further \(L_1 = 1, L_2 = 3\) and \(L_{18} = 5778\) are the only Lucas triangular numbers [12]. Note that the triangular number \(t_{n-1} = (n - 1)n/2\) is equal to the binomial coefficient \(\binom{n}{2}\). Therefore, it seems natural to search the binomial coefficients \(\binom{n}{k}\) in certain recurrences. Special cases of this question were handled by several authors, see, for example, [3].

Consider the binary recurrence \(U_m = AU_{m-1} + BU_{m-2}\) with arbitrary initial values \(U_0\) and \(U_1\). If \(\{V_m\}\) is the associate of \(\{U_m\}\) (i.e., the two
sequences have the same recurrence rule, further $V_0 = 2U_1 - AU_0$ and $V_1 = AU_1 + 2BU_0$, then their terms satisfy

$$V_n^2 - DU_n^2 = 4C(-B)^n,$$

(1.1)

where $D = A^2 + 4B$ and $C = U_1^2 - AU_0U_1 - BU_0^2$.

Fix $|B| = 1$. Replacing either $V_n$ or $U_n$ by $\binom{n}{k}$, (1.1) leads to the superelliptic equation

$$y^2 = Dn^4 - 2Dn^3 + Dn^2 \pm 16C.$$

The Magma [10] procedure IntegralQuarticPoints() may solve this equation. Hence if the lower index $k$ is 2 in $\binom{n}{k}$, then we are able to handle the problem for certain binary recurrences.

For the lower index $k = 3$ an algorithm was given in [16] to solve the equations

$$U_m = \binom{n}{3} \quad \text{and} \quad V_m = \binom{n}{3},$$

with the conditions $D > 0$, and $U_0 = 0$, $U_1 = 1$ (and $|B| = 1$). Illustrating the algorithm, all integer solutions to the equations

$$F_m = \binom{n}{3}, \quad L_m = \binom{n}{3} \quad \text{and} \quad P_m = \binom{n}{3}$$

were given in [16]. Here $P_m$ is a term of the Pell sequence.

Later, Szalay [15] treated the equations $F_m = \binom{n}{4}$, $L_m = \binom{n}{4}$, and Kovács [6] solved the analogous equation $P_m = \binom{n}{4}$. The more complicated problem $L_m = \binom{n}{5}$ was handled by Tengely [17].

In this paper, as a novelty, we do not fix the lower subscript $k$, but on the other hand we prescribe $n = 2^t$ with unknown non-negative integer $t$. Hence, for the Lucas numbers we study the diophantine equation

$$L_m = \binom{2^t}{k}.$$ 

The complete description of the result is given by the following theorem.

**Theorem 1.** The solutions to the diophantine equation

$$L_m = \binom{2^t}{k}$$

(1.2)

in non-negative integers $t$, $k \leq 2^{t-1}$, and $m$ are

$$(t,k,m) = (1,1,0), \ (2,1,3) \ \text{and} \ (a,0,1)$$

with non-negative integers $a$. 
LUCAS NUMBERS OF THE FORM \( \binom{2^t}{k} \)

2. Auxiliary results

Assume that \( p \) is a prime number. The \( p \)-adic order of a non-zero integer \( n \) is the largest positive integer exponent \( \nu \) of \( p \) such that \( p^\nu \) divides \( n \). As usual, let \( \nu \) be denoted by \( \nu_p(n) \). For the integer \( n = a_0 + a_1p + a_2p^2 + \cdots + a_dp^d \), \((0 \leq a_i < p)\), the digit sum function (in base \( p \)) is defined by
\[
s_p(n) = a_0 + a_1 + \cdots + a_d.
\]

In particular, Legendre [8] showed that
\[
\nu_p(n!) = n - s_p(n)/p - 1. \tag{2.1}
\]

**Lemma 1.** Assume that \( n \) and \( k \leq 2^n - 1 \) are positive integers. Then
\[
\nu_2 \left( \binom{2^n}{k} \right) = n - \nu_2(k).
\]

**Proof.** It is clear that \( \nu_2(2^n - j) = \nu_2(j) \) holds if \( 1 \leq j \leq 2^n - 1 \). Expanding the binomial coefficient we get
\[
\nu_2 \left( \binom{2^n}{k} \right) = \nu_2 \left( \frac{2^n(2^n - 1) \cdots (2^n - k + 1)}{k!} \right)
= \nu_2(2^n) + \nu_2(2^n - 1) + \cdots + \nu_2(2^n - (k - 1)) - \nu_2(k!)
= n + \nu_2((k - 1)! - \nu_2(k!)
= n - \nu_2(k).
\]

We note that Kummer [7] derived a result from Legendre’s theorem, which also proves the statement of the above lemma. Kummer’s theorem says that the \( p \)-adic valuation of the binomial coefficient \( \binom{a}{b} \) is equal to the number of carries when \( a - b \) is added to \( b \) in base \( p \).

Citing [9], here we present the 2-adic order of the Lucas numbers.

**Lemma 2.** If \( n \geq 0 \) is an integer, then
\[
\nu_2(L_n) = \begin{cases} 
0, & \text{if } n \equiv 1, 2 \pmod{3}, \\
1, & \text{if } n \equiv 0 \pmod{6}, \\
2, & \text{if } n \equiv 3 \pmod{6}.
\end{cases}
\]

**Lemma 3.** For any integer \( n \geq 0 \) we have \( L_n \not\equiv 6 \pmod{8} \).

**Proof.** Consider the Lucas numbers modulo 8. The sequence becomes periodic with length 12, and looking at the period, it leads immediately to the statement.

**Lemma 4.** A Lucas number \( L_n \) with odd subscript \( n \) is composed only of primes \( p \) satisfying \( p = 2 \) or \( p \equiv \pm 1 \pmod{5} \).
Proof. Although the proof comes straightaway from the well-know identity $L_n^2 - 5 F_n^2 = 4(-1)^n$, we simply refer to [13], last row of page 280. □

Lemma 5. Suppose that $a$, $b$ and $n$ are positive integers. Then

$$\left( \frac{an + bn}{an} \right) \equiv 0 \mod \left( \frac{bn + 1}{\gcd(a, bn + 1)} \right).$$

Proof. See Theorem 1.1 in [14]. □

Lemma 6. For $n \geq 1$ we have

$$\left( \frac{2^{n+1}}{2n} \right) \equiv 6 \mod 8.$$

Proof. It is obvious when $n = 1$. Therefore we may assume $n \geq 2$.

In case of $p = 2$ the Legendre formula (2.1) implies $\nu_2(2a!) = 2^a - 1$. Subsequently,

$$\nu_2 \left( \frac{2^{n+1}}{2n} \right) = \nu_2 \left( 2^{n+1} \left( \frac{2^{n+1}}{2n} \right) \right) = \nu_2(2^{n+1}) - 2\nu_2(2^n!) = 1,$$

hence it is sufficient to consider the odd ingredients of the binomial coefficient in the lemma. To do that, observe that $h(a) := 2^a/2^{2a-1}$ is an odd integer, and we need to see that $h(a) \equiv 3 \mod 8$ for $a \geq 2$. It is a direct consequence of Lemma 3.3 in the paper [4] by fixing there $p = 2$, $b = 3$, $t = 1$, $i = 0$, and $j = 1$. Finally, $h(n + 1)/h^2(n) \equiv 3/3^2 \equiv 3 \mod 8$ proves the lemma. □

3. Proof of Theorem 1

The statement is trivial for $k = 0$, and we obtain the infinite family of solutions $(t, k, m) = (a, 0, 1), a \geq 0$.

In the sequel, we assume $1 \leq k \leq 2^{t-1}$. Combining (1.2), Lemma 2, and Lemma 1, it provides

$$j = t - \nu_2(k),$$

where $j = 0, 1, 2$. Thus, $t - j = \nu_2(k)$, and, consequently, $k = 2^{t-j}s$ holds with some positive odd integer $s$. The condition $k = 2^{t-j}s \leq 2^{t-1}$ is fulfilled only if $j = 1$ or 2, and in these cases $s = 1$ necessarily holds. Hence $k = 2^{t-j}$ $(j = 1, 2)$. For our convenience put $a = t - j$. Then $k = 2^a$, and we distinguish two cases.

First let $j = 1$. Now we have the equation

$$L_m = \left( \frac{2a+1}{2a} \right)$$

to solve. Taking both sides of this equation modulo 8, Lemma 3 contradicts to Lemma 6 if $a \geq 1$. The remaining value $a = 0$ leads to the solution $(t, k, m) = (1, 1, 0)$. 

Now let $j = 2$. Clearly, by Lemma 2 we know that $m = 6\kappa + 3$. We have

$$L_m = \binom{2a^2}{2a},$$

and first assume that $a$ is even. The case $a = 0$ provides the solution $(t, k, m) = (2, 1, 3)$. Then we may suppose $a \geq 1$. Applying Lemma 5, it yields that

$$\binom{2a^2}{2a} = \binom{3 \cdot 2^a + 2^a}{3 \cdot 2^a} \equiv 0 \mod \left( \frac{2^a + 1}{\gcd(3, 2^a + 1)} \right).$$

The parity of $a$ guarantees that the denominator of the modulus is 1, i.e., the modulus is $2^a + 1$. Put $a = 2\ell$. Note that $\ell \geq 1$. Then we obtain

$$L_{6\kappa+3} = \binom{4^{\ell+1}}{4^\ell} \equiv 0 \mod 4^{\ell+1}.$$ 

This gives that $4^{\ell+1} \mid L_{6\kappa+3}$. By Lemma 4 we have

$$L_{6\kappa+3} = p_1p_2 \cdots p_n,$$

where $p_i$ are primes with $p_i = 2$ or $p_i \equiv \pm 1 \pmod{5}$ for $1 \leq i \leq n$. Hence every prime factor $p_j$ of $4^{\ell+1}$ ($\ell \geq 1$) has the form $p_j \equiv \pm 1 \pmod{5}$. Thus,

$$4^{\ell+1} = p_{i_1}p_{i_2} \cdots p_{i_t} \quad (3.1)$$

follows with $t \leq n$. Now reduce (3.1) modulo 5, and we arrive at a contradiction since $4^{\ell+1} \equiv 0 \text{ or } 2 \pmod{5}$, and at the same time $p_{i_1}p_{i_2} \cdots p_{i_t} \equiv 1 \text{ or } 4 \pmod{5}$.

Assume that $a$ is odd, and let $a = 2\ell + 1$ with a non-negative integer $\ell$. The case $\ell = 0$ does not provide a solution to (1.2). So we may assume $\ell \geq 1$. Now we get

$$\binom{2a^2}{2a} = \binom{2^a + 3 \cdot 2^a}{2a} \equiv 0 \pmod{3 \cdot 2^a + 1}$$

because trivially $\gcd(1, 3 \cdot 2^a + 1) = 1$. Thus,

$$3 \cdot 2^a + 1 = 6 \cdot 4^{\ell} + 1 \mid L_{6\kappa+3},$$

where the prime factors $p_j$ of $L_{6\kappa+3}$ again satisfy $p_j \equiv \pm 1 \pmod{5}$. A modulo 5 consideration of $6 \cdot 4^{\ell} + 1$, similarly to the previous case, leads to a contradiction.

The proof of Theorem 1 is complete.

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Niğde Ömer Halisdemir University, Art and Science Faculty, Mathematics Department, Niğde, Turkey
E-mail address: nirmak@ohu.edu.tr, irmaknurettin@gmail.com

J. Selye University, Department of Mathematics and Informatics, Komarno, Slovakia
E-mail address: laszlo.szalay.sopron@gmail.com