On a class of N(k)-mixed generalized quasi-Einstein manifolds

ARINDAM BHATTACHARYYA AND SAMPA PAHAN

ABSTRACT. The objective of the present paper is to study N(k)-mixed generalized quasi-Einstein manifolds. We prove the existence of these manifolds. Later we establish some curvature properties of N(k)-mixed generalized quasi-Einstein manifolds under certain conditions. In the last section, we give two examples of N(k)-mixed generalized quasi-Einstein manifolds.

1. Introduction

A Riemannian manifold \((M^n, g)\) with \(n \geq 2\) is said to be an Einstein manifold if the Ricci tensor \(S\) satisfies, on \(M\), the condition

\[ S(X, Y) = \frac{r}{n}g(X, Y), \]

where \(r\) denotes the scalar curvature of \((M^n, g)\). According to [1], the above equation is called the Einstein metric condition.

Chaki and Maity [3] introduced the concept of a quasi-Einstein manifold. A non-flat Riemannian manifold \((M^n, g)\), \(n \geq 2\), is said to be a quasi-Einstein manifold if the equality

\[ S(X,Y) = \alpha g(X,Y) + \beta \rho(X)\rho(Y) \]

is fulfilled on \(M\), where \(\alpha\) and \(\beta \neq 0\) are scalars, \(\rho\) is a non-zero 1-form such that \(g(X, \xi) = \rho(X)\) for all vector fields \(X\), and \(\xi\) is a unit vector field.

The notion of a mixed generalized quasi-Einstein manifold was introduced by Bhattacharyya and De in [2]. A non-flat Riemannian manifold is called
a mixed generalized quasi-Einstein manifold if its non-zero Ricci tensor $S$ of type $(0,2)$ satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y) + \delta [A(X)B(Y) + B(X)A(Y)],$$

where $\alpha, \beta, \gamma, \delta$ are non-zero scalars,

$$g(X, U) = A(X), \quad g(X, V) = B(X)$$

and $g(U, V) = 0$, $A, B$ being two non-zero 1-forms, and $U, V$ are unit vector fields corresponding to the 1-forms $A$ and $B$, respectively. This manifold is denoted by $MG(QE)_n$.

Let $R$ denote the Riemannian curvature tensor of a Riemannian manifold $M$. The $k$-nullity distribution $N(k)$ of the manifold $M$ is defined by (see [11])

$$N(k) : p \rightarrow N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)]Y\},$$

where $X, Y \in T_pM$ and $k$ is a smooth function. If the generators $U, V$ of a manifold $MG(QE)_n$ belong to $N(k)$, then we say that $(M^n, g)$ is a $N(k)$-mixed generalized quasi-Einstein manifold, and we denote it by $N(k)$-$MG(QE)_n$.

In 2007, Tripathi and Kim [12] studied $N(k)$-quasi-Einstein manifolds. They proved that an $n$-dimensional conformally flat quasi-Einstein manifold is an $N\left(\frac{a+b}{n-1}\right)$-quasi-Einstein manifold. Later many authors (see, for example, [10], [7], [13], [8]) have studied different types of $N(k)$-quasi-Einstein manifolds.

In this paper, we study the existence of $N(k)$-mixed generalized quasi-Einstein manifolds. Ricci-semi-symmetry, and the conharmonic and pseudo-projective curvature tensors of $N(k)$-$MG(QE)_n$ are characterized. We obtain Ricci recurrent, generalized Ricci recurrent and Ricci symmetric manifolds $N(k)$-$MG(QE)_n$. In the last section, we give two examples of $N(k)$-mixed generalized quasi-Einstein manifolds.

2. Existence of $N(k)$-mixed generalized quasi-Einstein manifolds

**Theorem 2.1.** Let $\mu, \lambda$ be nonzero scalars, let $U, W$ be vector fields on $M$, and let $Q : T_pM \rightarrow T_pM$ be a symmetric endomorphism such that $S(X, Y) = g(QX, Y)$. If in a conformally flat Riemannian manifold $(M^n, g)$, the Ricci tensor $S$ satisfies the relation

$$\mu S(Y, W)S(X, Z) + \lambda g(Y, W)g(X, Z) = [S(Y, Z)g(X, W) + g(Y, Z)S(X, W)] - [S(Y, W)g(X, Z) + S(X, Z)g(Y, W)],$$

(3)
and the condition

$$\lambda g(X, U)Y + \mu g(QX, U)QY = 0 \quad (4)$$

holds, then \((M^n, g)\) is a \(N(k)-\text{mixed generalized quasi-Einstein manifold}\).

**Proof.** Let \(U\) be the vector field defined by \(g(X, U) = P(X), X \in \chi(M)\). Taking \(X = W = U\) in (3), we get

\[
S(X, Y) = \alpha g(X, Y) + \beta T(X)T(Y) + \gamma P(X)P(Y) + \delta [T(X)P(Y) + P(X)T(Y)],
\]

where \(\alpha = -a/u, a = S(U, U), \beta = \mu/u, \gamma = \lambda/u, \delta = 1/u,\) and \(S(U, Z) = S(Z, U) = g(QZ, U) = P(QZ) = T(Z)\). Therefore, \((M^n, g)\) is a mixed generalized quasi-Einstein manifold.

If \((M^n, g)\) is conformally flat, then we have

\[
R(X, Y)Z = \frac{1}{n-2}\{g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y\} - \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\}. \quad (5)
\]

Taking \(Z = U\) in (5), for any \(W\) we obtain

\[
R(X, Y)U = \frac{1}{n-2}\{P(Y)QX - P(X)QY + S(Y, U)X - S(X, U)Y\} - \frac{r}{(n-1)(n-2)}\{P(Y)X - P(X)Y\}. \quad (6)
\]

Taking \(Z = U\) in (3), we obtain

\[
[S(Y, U)g(X, W) + g(Y, U)S(X, W)] - [S(X, U)g(Y, W) + S(Y, W)g(X, U)] = \mu S(Y, W)S(X, U) + \lambda g(Y, W)g(X, U),
\]

and thus

\[
g(S(Y, U)X + P(Y)QX - \mu T(X)QY - \lambda P(X)Y - S(X, U)Y - P(X)QY, W) = 0.
\]

Therefore from (4) we have

\[
S(Y, U)X - S(X, U)Y = P(X)QY - P(Y)QX.
\]

Substituting this in (6), we get

\[
R(X, Y)U = k(P(Y)X - P(X)Y),
\]

where \(k = -\frac{r}{(n-1)(n-2)}.\) Thus \(U \in N_p(k)\).

Suppose \(V\) is a vector field orthogonal to \(U\). Then we have \(V \in N_p(k)\). Hence \((M^n, g)\) is a \(N(k)-\text{mixed generalized quasi-Einstein manifold}\). \(\square\)
3. Ricci curvature, eigenvectors and associated scalars of manifolds $N(k)-MG(QE)_n$

From (1), we deduce that
\[ S(U,U) = \alpha + \beta, \quad S(V,V) = \alpha + \gamma, \quad S(U,V) = \delta = S(V,U) \]
since $g(U,V) = 0$.

It is well known that $S(X,X)$ is the Ricci curvature in the direction of a unit vector field $X$. Now if $X$ is a unit vector field in the section spanned by $U$ and $V$, then we have
\[ 1 = g(X,X) = g(aU + bV,aU + bV) = a^2 + b^2 \]
since $g(U,V) = 0$ and $g(U,U) = 1, \quad g(V,V) = 1$.

Thus we can formulate the following result.

**Theorem 3.1.** In $N(k)-MG(QE)_n$, the Ricci curvature in the direction of $U$ is $\alpha + \beta$, and in the direction of $V$ is $\alpha + \gamma$. The Ricci curvature in all other directions of the section of $U$ and $V$ is
\[ \alpha + \beta A(X)A(Y) + \gamma B(X)(Y) + 2\delta A(X)B(X). \]

Let $(M^n, g)$ be a $N(k)$-mixed generalized quasi-Einstein manifold. Since $U,V \in N_p(k)$, we have
\[ g(R(X,Y)U,W) = k\{A(Y)g(X,W) - A(X)g(Y,W)\}. \]

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the tangent space $T_pM$ at any point $p \in M$. Putting $X = W = e_i$ and summing over $i, \quad 1 \leq i \leq n$, we obtain
\[ S(Y,U) = k(n-1)A(X). \quad (7) \]

Similarly,
\[ S(Y,V) = k(n-1)B(X). \quad (8) \]

From (1), we get
\[ S(X,U) = (\alpha + \beta)A(X) + \delta B(X), \quad (9) \]
\[ S(X,V) = (\alpha + \gamma)B(X) + \delta A(X). \quad (10) \]

Subtracting (8) from (7) and (10) from (9), we see that
\[ k(n-1) = \alpha + \beta - \delta, \quad (11) \]
\[ k(n-1) = \alpha + \gamma - \delta. \quad (12) \]

Hence, adding (11) and (12), we obtain
\[ k = \frac{2\alpha + \beta + \gamma - 2\delta}{2(n-1)}. \]
Therefore,
\[ S(X, U) = \frac{2\alpha + \beta + \gamma - 2\delta}{2} g(X, U) \]
and
\[ S(X, V) = \frac{2\alpha + \beta + \gamma - 2\delta}{2} g(X, V). \]

Consequently, \( U \) and \( V \) are eigenvectors corresponding to the eigenvalue \((2\alpha + \beta + \gamma - 2\delta)/2\).

4. Curvature tensors of manifolds \( N(k)\)-MG\((QE)\)_n

Let \((M, g)\) be a Riemannian manifold of dimension \(n\). The conharmonic curvature tensor is defined by

\[ \bar{C}(X, Y)Z = R(X, Y)Z - \frac{1}{n-2} \{ S(Y, Z)X - S(X, Z)Y \\
+ g(Y, Z)QX - g(X, Z)QY \}, \tag{13} \]

where \(X, Y, Z \in \chi(M)\) and \(Q\) is the Ricci operator.

The pseudo-projective curvature tensor is defined by (see [9])

\[ \bar{P}(X, Y)Z = aR(X, Y)Z + b\{ S(Y, Z)X - S(X, Z)Y \} \\
- \frac{r}{n} \left[ \frac{a}{n-1} + b \right] \{ g(Y, Z)X - g(X, Z)Y \}, \tag{14} \]

where \(X, Y, Z \in \chi(M)\), \(a, b \neq 0\) are constants, \(Q\) is the Ricci operator, and \(r\) is the scalar curvature.

Now we establish the following theorems.

**Theorem 4.1.** An \(n\)-dimensional \(N(k)\)-mixed generalized quasi-Einstein manifold \(M\) satisfies the condition \(\bar{C}(U, Y) \cdot S = 0\) if and only if

\[ k(n - 2) [n(\alpha + \beta) - r] - [n(\alpha + \beta)^2 + (n - 1)\delta^2] \\
- \alpha(\gamma + r) - \gamma(\gamma + \delta) - \beta(\beta + \alpha) = 0, \]

and the condition \(\bar{C}(V, Y) \cdot S = 0\) if and only if

\[ k(n - 2) [n(\alpha + \gamma) - r] - [n(\alpha + \gamma)^2 + (n - 1)\delta^2] \\
- \alpha(\beta + r) - \beta(\beta + \delta) - \gamma(\gamma + \alpha) = 0, \]

where \(r\) is the scalar curvature.

**Proof.** Since \(\bar{C}(U, Y) \cdot S = 0\), we have

\[ S(\bar{C}(U, Y)Z, W) + S(Z, \bar{C}(U, Y)W) = 0. \tag{15} \]

Then, by (2) and (13), we have that

\[ k[g(Y, Z)S(U, W) - g(U, Z)S(Y, W) + g(Y, W)S(U, Z) - g(U, W)S(Y, Z)] \]
The theorem has been proved. □

Similarly, we get that \( \bar{\alpha} \)

From (16), we get

\[
\begin{align*}
\frac{1}{n-2}[g(Y, Z)S(QU, W) - g(U, Z)S(QY, W)] + g(Y, W)S(QU, Z) - g(U, W)S(QY, Z)] = 0.
\end{align*}
\]

Putting \( W = U \), we get

\[
\begin{align*}
k[g(Y, Z)S(U, U) - g(U, Z)S(Y, U) + g(Y, U)S(U, Z) & - g(U, U)S(Y, Z)] - \frac{1}{n-2}[g(Y, Z)S(QU, U) - g(U, Z)S(QY, U)] \\
& + g(Y, U)S(QU, Z) - g(U, U)S(QY, Z)] = 0.
\end{align*}
\]

From (1), we have

\[
QX = \alpha X + \beta A(X)U + \gamma B(X)V + \delta[A(X)V + B(X)U].
\] (17)

From (16), we get

\[
\begin{align*}
k[g(Y, Z)(\alpha + \beta) - g(U, Z)(\alpha g(Y, U) + \beta A(Y)\delta B(Y))] & + g(Y, U)(\alpha g(Z, U) + \beta A(Z) + \delta B(Z)) - S(Y, Z)] \\
& - \frac{1}{n-2}[g(Y, Z)S(\alpha U + \beta U + \delta V, U) - g(U, Z)S(\alpha Y + \beta A(Y)U + \gamma B(Y)V) \\
& + \delta[A(Y)V + B(Y)U], U)] = 0.
\end{align*}
\] (18)

Let \( \{e_1, e_2, ..., e_n\} \) be an orthonormal basis of the tangent space \( T_pM \) at any point \( p \in M \). Putting \( Y = Z = e_i \) and summing over \( i, 1 \leq i \leq n \), we obtain

\[
k(n-2)[n(\alpha + \beta) - r] - [n(\alpha + \beta)^2 + (n-1)\delta^2 - \alpha(\gamma + r) - \gamma(\gamma + \delta) - \beta(\beta + \alpha)] = 0.
\]

Similarly, we get that \( \bar{C}(V, Y) \cdot S = 0 \) if and only if

\[
k(n-2)[n(\alpha + \gamma) - r] - [n(\alpha + \gamma)^2 + (n-1)\delta^2 - \alpha(\beta + r) - \beta(\beta + \delta) - \gamma(\gamma + \alpha)] = 0,
\]

The theorem has been proved. □

**Theorem 4.2.** A \( n \)-dimensional \( N(k) \)-mixed generalized quasi-Einstein manifold \( M \) satisfies the condition \( \bar{P}(U, Y) \cdot S = 0 \) if and only if either

\[
ak - \frac{r}{n} \left( \frac{\alpha}{n-1} + b \right) = 0 \text{ or } n(\alpha + \beta) = r,
\]

and the condition \( \bar{P}(V, Y) \cdot S = 0 \) if and only if either

\[
ak - \frac{r}{n} \left( \frac{\alpha}{n-1} + b \right) = 0 \text{ or } n(\alpha + \gamma) = r.
\]

**Proof.** Since \( \bar{P}(U, Y) \cdot S = 0 \), we have

\[
S(\bar{P}(U, Y)Z, W) + S(Z, \bar{P}(U, Y)W) = 0.
\] (19)
By (2) and (14), we get
\[
\left[ ak - \frac{r}{n} \left( a + b \right) \right] \left[ g(Y, Z)S(U, W) - g(U, Z)S(Y, W) \\
+ g(Y, W)S(U, Z) - g(U, W)S(Y, Z) \right] = 0.
\]

Putting \( W = U \), we obtain
\[
\left[ ak - \frac{r}{n} \left( a + b \right) \right] \left[ g(Y, Z)(\alpha + \beta) - g(U, Z)[\alpha g(Y, U) + \beta A(Y) \\
+ \delta B(Y)] + g(Y, U)[\alpha g(Z, U) + \beta A(Z) + \gamma B(Z)] - S(Y, Z) \right] = 0.
\]

(20)

Let \( \{e_1, e_2, ..., e_n\} \) be an orthonormal basis of the tangent space \( T_pM \) at any point \( p \in M \). Putting \( Y = Z = e_i \) and summing over \( i \), \( 1 \leq i \leq n \), we obtain
\[
ak - \frac{r}{n} \left( a + b \right) = 0 \text{ or } n(\alpha + \beta) = r.
\]

Similarly, we get that \( \bar{P}(V, Y) \cdot S = 0 \) if and only if \( ak - \frac{r}{n} \left( a + b \right) = 0 \) or \( n(\alpha + \gamma) = r \).

This completes the proof. \( \square \)

5. Ricci-recurrent manifolds \( N(k)-MG(QE)_n \)

A manifold \( N(k)-MG(QE)_n \) is said to be Ricci-recurrent if its Ricci tensor \( S \) of type \( (0, 2) \) satisfies the condition
\[
(\nabla_X S)(Y, Z) = L(X)S(Y, Z),
\]
where \( L \) is the nonzero 1-form such that \( L(X) = g(X, \xi) \) holds, \( \xi \) being the associated vector field of the 1-form \( L \).

A manifold \( N(k)-MG(QE)_n \) is said to be generalized Ricci-recurrent if its Ricci tensor \( S \) of type \( (0, 2) \) satisfies the condition
\[
(\nabla_X S)(Y, Z) = F(X)S(Y, Z) + G(X)g(Y, Z),
\]
where \( F, G \) are the nonzero 1-forms such that \( F(X) = g(X, \xi_1) \), \( G(X) = g(X, \xi_2) \), and \( \xi_1, \xi_2 \) are associated vector fields of the 1-forms \( F, G \), respectively.

We prove the following proposition.

**Proposition 5.1.** Let \( F, G \) be nonzero 1-forms. In a generalized Ricci-recurrent manifold \( N(k)-MG(QE)_n \), the following statements are true.

(i) If \( U \) is a parallel vector field, then \( X(\alpha + \beta) = (\alpha + \beta)F(X) + G(X) \).

(ii) If \( V \) is a parallel vector field, then \( X(\alpha + \gamma) = (\alpha + \gamma)F(X) + G(X) \).

**Proof.** Putting \( Y = Z = U \) in (22), we get
\[
(\nabla_X S)(U, U) = (\alpha + \beta)F(X) + G(X).
\]
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On the other hand, we have
\[(\nabla_X S)(U,U) = X(\alpha + \beta) - 2\delta S(\nabla_X U, U),\]
i.e.,
\[2[(\alpha + \beta)A(\nabla_X U) + \delta B(\nabla_X U)] = X(\alpha + \beta) - (\alpha + \beta)F(X) - G(X).\]

Since \(U\) is parallel vector field, \(\nabla_X U = 0\). Then from the above we get
\[X(\alpha + \beta) = (\alpha + \beta)F(X) + G(X).\]
Similarly we can show that if \(V\) is a parallel vector field, then
\[X(\alpha + \gamma) = (\alpha + \gamma)F(X) + G(X).\]
The proof is complete. \(\square\)

From the previous proposition we have the following corollary.

**Corollary 5.1.** Let \(L\) be a nonzero 1-form. In a Ricci-recurrent manifold \(N(k)\)-MG\((QE)n\), the following statements hold.

(i) If \(U\) is parallel vector field, then \(d(\alpha + \beta)(X) = (\alpha + \beta)L(X)\).

(ii) If \(V\) is parallel vector field, then \(d(\alpha + \gamma)(X) = (\alpha + \gamma)L(X)\).

6. Ricci-symmetric manifolds \(N(k)\)-MG\((QE)n\)

A Riemannian manifold \((M^n, g)\) is said to be Ricci-semi-symmetric if the relation \(R(X,Y)\cdot S = 0\) holds, where \(R(X,Y)\) is the curvature operator and \(S\) is the Ricci tensor of type \((0,2)\).

**Theorem 6.1.** An \(N(k)\)-mixed generalized quasi-Einstein manifold satisfies the relations \(R(U,Y)\cdot S = 0\) and \(R(V,Y)\cdot S = 0\) if and only if \(k = 0\).

**Proof.** Let \((M^n, g)\) be a Ricci-semi-symmetric manifold \(N(k)\)-MG\((QE)n\). Then we have
\[S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0.\] (23)

Putting \(X = V\) in (23), we obtain
\[k\{g(Y,Z)S(V,W) - B(Z)S(Y,W) + g(Y,W)S(Z,V) - B(W)S(Z,Y)\} = 0.\] (24)

Putting \(W = V\), we get
\[k\{(\alpha + \gamma)g(Y,Z) - \delta A(Y)B(Z) + \delta A(Y)B(Z) - S(Y,Z)\} = 0.\]
Hence either \(k = 0\) or
\[\{g(Y,Z) - \delta A(Y)B(Z) + dA(Y)B(Z) - S(Y,Z)\} = 0.\]
If \(k \neq 0\), then in the second case the manifold becomes an \(N(k)\)-mixed quasi-Einstein manifold (see [6]) which is not possible. Therefore we must have \(k = 0\).

Conversely, suppose \(k = 0\). Then we obtain that \(R(V,Y)\cdot S = 0\).
Similarly, we get that $R(U,Y) \cdot S = 0$ if and only if $k = 0$, and the proof is complete. □

A manifold $N(k)-M(GQ)_n$ is said to be Ricci-symmetric if its Ricci tensor $S$ of type $(0,2)$ satisfies the condition
\[(\nabla_X S)(Y,Z) = 0\] (25)
for all $X,Y,Z \in \chi(M)$.

**Proposition 6.1.** If a manifold $N(k)-MG(QE)_n$ with constant associated scalar is Ricci-symmetric with Levi-Civita connection $\nabla$, and $U$ is a parallel vector field, then
\[b(\nabla_X A)(Y) + d(\nabla_X B)(Y) = 0.\]

**Proof.** First, putting $Z = U$ in (25), where $U$ is a parallel vector field, we have
\[\beta(\nabla_X A)(Y) + \delta(\nabla_X B)(Y) = 0.\]
Similarly, if $V$ is a parallel vector field and $M$ is Ricci-symmetric manifold $N(k)-MG(QE)_n$, then we can show that
\[\gamma(\nabla_X B)(Y) + \delta(\nabla_X A)(Y) = 0,\]
which completes the proof. □

**Corollary 6.1.** If a manifold $N(k)-MG(QE)_n$ with constant associated scalar is Ricci-symmetric with Levi-Civita connection $\nabla$, and $V$ is a parallel vector field, then
\[\gamma(\nabla_X B)(Y) + \delta(\nabla_X A)(Y) = 0.\]

### 7. Examples of manifolds $N(k)-MG(QE)_n$

**Example 7.1.** Let us consider a Riemannian metric $g$ on $R^4$ determined by
\[ds^2 = g_{ij}dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],\]
where $i,j = 1,2,3,4$ and $p = k^{-2}e^{x^1}$, $k$ is constant. Then the only non-vanishing components of Christoffel symbols, the curvature tensors, and the Ricci tensors are
\[
\begin{align*}
\Gamma^1_{22} &= \Gamma^1_{33} = \Gamma^1_{44} = -\frac{p}{1 + 2p}, \\
\Gamma^1_{11} &= \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{p}{1 + 2p}, \\
R_{1221} &= R_{1331} = R_{1441} = \frac{p}{1 + 2p}, \\
R_{2332} &= R_{2442} = R_{3443} = \frac{p^2}{1 + 2p}, \\
R_{11} &= \frac{3p}{(1 + 2p)^2}, \quad R_{22} = R_{33} = R_{44} = \frac{p}{(1 + 2p)}.\end{align*}
\]
Let us consider the associated scalars $\alpha, \beta, \gamma, \delta$ defined by
\[
\alpha = \frac{p}{(1 + 2p)^2}, \quad \beta = \frac{2p}{(1 + 2p)^3}, \quad \gamma = \frac{p}{(1 + 2p)^3}, \quad \delta = -\frac{p}{2(1 + 2p)^2},
\]
and the 1-forms
\[
A_i(x) = B_i(x) = \begin{cases} \sqrt{1 + 2p} & \text{if } i = 1, \\ 0 & \text{otherwise}, \end{cases}
\]
where generators are unit vector fields. Then we have
(i) $R_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta [A_1 B_1 + A_1 B_1]$, \\
(ii) $R_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta [A_2 B_2 + A_2 B_2]$, \\
(iii) $R_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta [A_3 B_3 + A_3 B_3]$, \\
(iv) $R_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta [A_4 B_4 + A_4 B_4]$.

Since all the cases (i)–(iv) are trivial, we can say that
\[
R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j + \delta [A_i B_j + A_j B_i], \quad i, j = 1, 2, 3, 4.
\]
So, $(R^4, g)$ is a mixed generalized quasi-Einstein manifold with non-zero and non-constant scalar curvature. We can say that $(M^4, g)$ is an $N(\frac{p(2+p)}{3(1+2p)^2})$ mixed generalized quasi-Einstein manifold.

**Example 7.2.** Let us consider a Riemannian metric $g$ on $R^4$ by
\[
ds^2 = g_{ij} dx^i dx^j = (dx^1)^2 + e^{x^1+x^2} (dx^2)^2 \\
+ e^{x^1+x^3} (dx^3)^2 + e^{x^1+x^4} (dx^4)^2, \quad i, j = 1, 2, 3, 4.
\]
Then the only non-vanishing components of Christoffel symbols, the curvature tensors, and the Ricci tensors are
\[
\Gamma^1_{22} = -\frac{1}{2} e^{x^1+x^2}, \quad \Gamma^1_{33} = -\frac{1}{2} e^{x^1+x^3}, \quad \Gamma^4_{44} = -\frac{1}{2} e^{x^1+x^4},
\]
\[
\Gamma^2_{22} = \Gamma^3_{33} = \Gamma^4_{44} = \frac{1}{2} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14},
\]
\[
R_{1221} = \frac{1}{4} e^{x^1+x^2}, \quad R_{1331} = \frac{1}{4} e^{x^1+x^3}, \quad R_{1441} = \frac{1}{4} e^{x^1+x^4},
\]
\[
R_{2332} = \frac{1}{4} e^{2x^1+x^2+x^3}, \quad R_{2442} = \frac{1}{4} e^{2x^1+x^2+x^4}, \quad R_{3443} = \frac{1}{4} e^{2x^1+x^3+x^4},
\]
\[
R_{11} = \frac{3}{4}, \quad R_{22} = \frac{3}{4} e^{x^1+x^2}, \quad R_{33} = \frac{3}{4} e^{x^1+x^3}, \quad R_{44} = \frac{3}{4} e^{x^1+x^4}.
\]
Let us consider the associated scalars $\alpha, \beta, \gamma, \delta$ defined by
\[
\alpha = \frac{3}{4}, \quad \beta = e^{x^1}, \quad \gamma = \frac{2}{e^{x^1}}, \quad \delta = -\frac{2}{\sqrt{2}}.
\]
and the 1-forms
\[ A_i(x) = \begin{cases} \sqrt{2}e^{x_1} & \text{if } i = 1, \\ 0 & \text{otherwise,} \end{cases} \quad B_i(x) = \begin{cases} \sqrt{e^{x_1}} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases} \]
where generators are unit vector fields. Then we have
\begin{align*}
(i) \quad R_{11} &= \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta [A_1 B_1 + A_1 B_1], \\
(ii) \quad R_{22} &= \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta [A_2 B_2 + A_2 B_2], \\
(iii) \quad R_{33} &= \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta [A_3 B_3 + A_3 B_3], \\
(iv) \quad R_{44} &= \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta [A_4 B_4 + A_4 B_4].
\end{align*}
Since all the cases (i)-(iv) are trivial, we can say that
\[ R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j + \delta [A_i B_j + A_j B_i], \quad i, j = 1, 2, 3, 4. \]
So, in this case \((R^4, g)\) is a mixed generalized quasi-Einstein manifold. We can easily see that \((M^4, g)\) is an \(N(2\sqrt{2}e^{x_1} + 4 \sqrt{e^{x_1}} + 3 \sqrt{2}e^{x_1} + 12 \sqrt{2}e^{x_1})\)-mixed generalized quasi-Einstein manifold.

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References


Department of Mathematics, Jadavpur University, Kolkata-700032, India

E-mail address: bhattchar1968@yahoo.co.in

E-mail address: sampapahan25@gmail.com