A new approach to nearly compact spaces

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ABSTRACT. Using the covers formed by pre-open sets, we introduce and study the notion of po-compactness in topological spaces. The notion of po-compactness is weaker than that of compactness but stronger than semi-compactness. It is observed that po-compact spaces are the same as nearly compact spaces. However, we find new characterizations to near compactness, when we study it in the sense of po-compactness.

1. Introduction

Unless otherwise mentioned, $X$ stands for the topological space $(X, \mathcal{P})$. $Int_X(A)$ or $Int(A)$ (respectively, $Cl_X(A)$ or $Cl(A)$) denotes the interior (respectively, closure) of a subset $A$ in a topological space $X$.

Generalizing the concept of open sets, Levine [7] introduced the notion of semi-open sets: a subset $A$ of a topological space $X$ is semi-open if there exists an open set $G$ such that $G \subset A \subset Cl(G)$. One more generalization of open sets is the notion of pre-open sets (see Mashhour et al. [8]) introduced by Corson and Michael [3] under the name locally dense sets: a subset $A$ of a topological space $X$ is called locally dense if there exists an open set $U$ such that $A \subset U \subset Cl(A)$. The complement of a pre-open set is called a pre-closed set (see [8]). The concepts of semi-open sets and pre-open sets are independent. Several covering properties have been introduced and studied using covers formed by semi-open and pre-open sets (see, for example, [2, 6, 9, 8, 11, 16, 17]). A cover formed by semi-open sets is called an s-cover (see [16]). A topological space $X$ is called s-compact (see [16]) if each s-cover of $X$ has a finite subcover. However, s-compactness has been widely studied under the name semi-compactness by Dorsett [4]. As semi-compactness is...
stronger than compactness, and the term semi-compact delineates a notion weaker than compactness, we retain the term s-compactness due to Prasad and Yadav [16] to mean the notion of semi-compactness due to Dorsett [4] as well, and henceforth, by semi-compactness, we mean the notion introduced by Mukharjee et al. [11]. Mashhour et al. [9] introduced a compact like notion called strong compactness: a topological space $X$ is called strongly compact if each cover of $X$ by pre-open sets of $X$ has a finite subcover.

For a topological space $(X, \mathcal{P})$ and a subset $A \subset X$, we write $(A, \mathcal{P}_A)$ to denote the subspace on $A$ of $(X, \mathcal{P})$. We also write $SO(X)$ (respectively, $PO(X)$) to denote the collection of all semi-open (respectively, pre-open) sets of $X$. Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers.

2. Po-compactness

We begin by recalling that a subset $A$ of a topological space $X$ is called regularly open if $A = \text{Int} (\text{Cl}(A))$. So if $G$ is open in $X$, then $\text{Int} (\text{Cl}(G))$ is regularly open in $X$.

We agree to mean by an open collection and pre-open collection a collection consisting, respectively, of open sets and pre-open sets of a topological space $X$. An open (respectively, a pre-open) collection $A$ of subsets of $X$ such that $\bigcup_{A \in \mathcal{A}} A = X$ is called an open cover (respectively, a pre-open cover) of $X$. The terms regularly open collection and regularly open cover are apparent. A collection $\mathcal{A}$ of subsets of $X$ is called a weak cover of $X$ if $\text{Cl} (\bigcup_{A \in \mathcal{A}} A) = X$.

**Definition 2.1** (Singal and Mathur [18]). A topological space $X$ is called nearly compact if, for each open cover $\mathcal{U}$ of $X$, there exists a finite subcollection $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup \{\text{Int} (\text{Cl}(V)) \mid V \in \mathcal{V}\} = X$.

**Definition 2.2** (Alexandroff and Urysohn [1], see also [5] or [15]). A Hausdorff topological space is called $H$-closed if the space is closed in every Hausdorff topological space containing it as a subspace.

It is seen that a Hausdorff topological space $X$ is $H$-closed if and only if each open cover $\mathcal{U}$ of $X$ has a finite subcollection $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup \{\text{Cl}(V) \mid V \in \mathcal{V}\} = X$.

Dropping the Hausdorffness from the notion of $H$-closedness, we get a type of covering notion called quasi $H$-closed spaces (see [15]).

**Definition 2.3** (Thompson [19]). A topological space $X$ is called $S$-closed if for each $s$-cover $\mathcal{U}$ of $X$, there exists a finite subcollection $\mathcal{V} \subset \mathcal{U}$ such that $\{\text{Cl}(V) \mid V \in \mathcal{V}\}$ covers $X$.
Definition 2.4 (Mukharjee et al. [11]). Let $\mathcal{U}$ be a collection of open sets of $X$. Then the collection

$$\mathcal{V} = \{ V \mid U \in \mathcal{U}, U \subset V \subset Cl(U), V \neq Cl(U) \text{ when } Cl(U) \notin \mathcal{P} \}$$

is called a semi-open super-collection of $\mathcal{U}$.

We note that $\mathcal{V}$ is a cover of $X$ if $\mathcal{U}$ is a cover of $X$. In this case, $\mathcal{V}$ is called a semi-open super-cover of the open cover $\mathcal{U}$.

Definition 2.5 (Mukharjee et al. [11]). A topological space $X$ is called semi-compact if each open cover of $X$ has a finite semi-open super-cover.

We now introduce the following definitions.

Definition 2.6. Let $\mathcal{S}$ be a pre-open collection of $X$. If, for each $A \in \mathcal{S}$, there exists an open set $U$ such that $A \subset U \subset Cl(A)$, then the collection

$$\mathcal{U} = \{ U \mid A \in \mathcal{S}, A \subset U \subset Cl(A) \}$$

is said to be an open super-collection of $\mathcal{S}$.

Note that there always exists an open super-collection of a pre-open collection of a topological space $X$. Let us also note that $\mathcal{U}$ is a cover of $X$ if $\mathcal{S}$ is a cover of $X$. In this case, $\mathcal{U}$ is said to be an open super-cover of the pre-open cover $\mathcal{S}$.

Definition 2.7. A topological space $X$ is said to be po-compact if each pre-open cover of $X$ has a finite open super-cover.

Let $\mathcal{U}$ be a finite open super-cover of a pre-open collection of a topological space $X$. For each $U \in \mathcal{U}$, there exists a pre-open set $A \in \mathcal{S}$ such that $A \subset U \subset Cl(A)$. Thus we have a finite subcollection $\{ A \mid U \in \mathcal{U}, A \subset U \subset Cl(A) \}$ of $\mathcal{S}$ corresponding to $\mathcal{U}$.

It is easy to see that compact and hence strongly compact spaces, and $s$-compact spaces are po-compact spaces. However, a po-compact space need not be a compact space, and a semi-compact space need not be a po-compact space.

Example 1. Let $\mathcal{P} = \{ (-\infty, n) \mid n \in \mathbb{N} \}$. The topological space $(\mathbb{R}, \mathcal{P})$ is po-compact, but not compact.

We summarize the implication relations of po-compactness with compactness and compact like notions in the following diagram for better understanding its position.
In the figure, “P → Q” stands to mean “P implies Q” and “P ∉ Q” stands to mean “P does not imply Q”.

**Theorem 2.1.** A space X is po-compact if and only if it is nearly compact.

**Proof.** The necessity follows directly from the fact that open sets in a topological space X are pre-open.

Conversely, let X be a nearly compact space and let 𝓨 be a pre-open cover of X. For each A ∈ 𝓨, there exists an open set G such that A ⊂ G ⊂ Cl(A).

Since 𝓨 is a cover of X, the collection 𝓨 = {G ∈ 𝓡 | A ⊂ G ⊂ Cl(A), A ∈ 𝓨} is an open cover of X. By near compactness of X, the collection 𝓨 has a finite subcollection 𝓨ₙ = {Gₖ ∈ 𝓨 | k ∈ {1, 2, ..., n}} such that \( \bigcup_{k=1}^{n} \text{Int}(\text{Cl}(G_k)) = X \). For each k ∈ {1, 2, ..., n}, there exists an Aₖ ∈ 𝓨 such that Aₖ ⊂ Gₖ ⊂ Cl(Aₖ). It means that Cl(Aₖ) = Cl(Gₖ) which implies that Aₖ ⊂ Int(Cl(Gₖ)) = Int(Cl(Aₖ)) ⊂ Cl(Aₖ) for each k ∈ {1, 2, ..., n}. So \( \{\text{Int}(\text{Cl}(G_k)) | G_k ∈ 𝓨_n, k ∈ \{1, 2, ..., n\}\} \) is a finite open super-cover of 𝓨.

**Lemma 2.1.** If A is pre-open, then \( \text{Int}(\text{Cl}(A)) \) is regularly open.

**Proof.** If A is a pre-open set, then there exists an open set G such that Cl(A) = Cl(G). So \( \text{Int}(\text{Cl}(A)) \) is regularly open.

**Theorem 2.2.** A topological space X is nearly compact if and only if each pre-open cover 𝓨 of X has a finite regularly open super-cover \( \{\text{Int}(\text{Cl}(A)) | A ∈ 𝓨\} \), where 𝓨 is a finite subcollection of 𝓨.

**Proof.** Suppose that 𝓨 is a pre-open cover of a nearly compact topological space X. By near compactness of X, the cover 𝓨 has a finite open super-cover 𝓨. For each G ∈ 𝓨, we get a pre-open set A ∈ 𝓨 such that A ⊂ G ⊂ Cl(A) and hence A ⊂ G ⊂ Int(Cl(A)) ⊂ Cl(A). Thus we get a finite subcollection 𝓦 = \( \{A ∈ 𝓨 | G ∈ 𝓨, A ⊂ G ⊂ Cl(A)\} \) of 𝓨. As 𝓨 is a cover of X, \( \{\text{Int}(\text{Cl}(A)) | A ∈ 𝓨\} \) is also a cover of X. By Lemma 2.1, \( \text{Int}(\text{Cl}(B)) \) is regularly open for each B ∈ 𝓨. So 𝓨 is a finite subcollection of 𝓨 such that \( \{\text{Int}(\text{Cl}(B)) | B ∈ 𝓨\} \) is a regularly open super-cover of the pre-open cover 𝓨 of X.
Conversely, since $\text{Int}(\text{Cl}(A))$ is open and $A \subset \text{Int}(\text{Cl}(A)) \subset \text{Cl}(A)$ for each $A \in \mathcal{B}$, the collection $\{\text{Int}(\text{Cl}(A)) \mid A \in \mathcal{B}\}$ is a finite open super-cover of $\mathcal{A}$. So $X$ is nearly compact. \hspace{1cm} \Box

**Corollary 2.1.** If $\mathcal{F}$ is a collection of pre-closed sets such that $\bigcap_{F \in \mathcal{F}} F = \emptyset$ in a nearly compact space $X$, then there exists a finite subcollection $\mathcal{E}$ of $\mathcal{F}$ such that $\bigcap_{E \in \mathcal{E}} \text{Cl}(\text{Int}(E)) = \emptyset$.

**Proof.** Since $\mathcal{F}$ is a collection of pre-closed sets satisfying $\bigcap_{F \in \mathcal{F}} F = \emptyset$, the collection $\{X - F \mid F \in \mathcal{F}\}$ is a pre-open cover of $X$. By Theorem 2.2, we get a finite subcollection $\{X - F_1, X - F_2, \ldots, X - F_n\}$ of $\mathcal{F}$ such that $\bigcup_{k=1}^n \text{Int}(\text{Cl}(X - F_k)) = X$. Hence $\bigcap_{k=1}^n \text{Cl}(\text{Int}(F_k)) = \emptyset$. \hspace{1cm} \Box

**Theorem 2.3.** Each pre-open cover of a nearly compact space has a finite pre-open weak cover.

**Proof.** Let $X$ be a nearly compact space and let $\mathcal{A}$ be a pre-open cover of $X$. By near compactness of $X$, the cover $\mathcal{A}$ has a finite open super-cover $\mathcal{B}$. For each $G \in \mathcal{B}$, there exists an $A \in \mathcal{A}$ such that $A \subset \text{Int}(G)$ which implies that $\text{Cl}(A) = \text{Cl}(G)$. So it follows that $\mathcal{B} = \{A \mid A \subset \text{Int}(G) \subset \text{Cl}(A), G \in \mathcal{B}\}$ is a finite pre-open weak cover of $X$. \hspace{1cm} \Box

**Theorem 2.4.** In a topological space $X$, the following statements are equivalent.

(a) $X$ is nearly compact.

(b) Each pre-open cover $\mathcal{A}$ of $X$ has a finite subcollection $\mathcal{B}$ such that $\{\text{Int}(\text{Cl}(B)) \mid B \in \mathcal{B}\}$ covers $X$.

(c) Each family $\mathcal{F}$ of pre-closed sets has nonempty intersection if $\bigcap \{\text{Int}(\text{Cl}(E)) \mid E \in \mathcal{E}\} \neq \emptyset$ for each finite subcollection $\mathcal{E}$ of $\mathcal{F}$.

**Proof.** (a)$\Rightarrow$(b) follows from Theorem 2.2.

(b)$\Rightarrow$(c). Let $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$ be a collection of pre-closed sets of $X$ such that for each finite subfamily $\mathcal{E}$ of $\mathcal{F}$, $\bigcap \{\text{Cl}(\text{Int}(E)) \mid E \in \mathcal{E}\} \neq \emptyset$. If possible, let $\bigcap_{F \in \mathcal{F}} F = \emptyset$. Then $\mathcal{G} = \{X - F_\alpha \mid \alpha \in A\}$ is a pre-open cover of $X$. By (b), we get a finite subcollection $\{X - F_{\alpha_k} \mid k \in \{1, 2, \ldots, n\}\}$ of $\mathcal{G}$ such that $\{\text{Int}(\text{Cl}(X - F_{\alpha_k})) \mid k \in \{1, 2, \ldots, n\}\}$ covers $X$. It means that $X - \bigcup_{k=1}^n \text{Int}(\text{Cl}(X - F_{\alpha_k})) = \emptyset$ which in turn implies that $\bigcap_{k=1}^n \text{Cl}(\text{Int}(F_{\alpha_k})) = \emptyset$, a contradiction to our assumption.

(c)$\Rightarrow$(a). Let $X$ be a topological space satisfying (c). Suppose for contradiction that $X$ is not nearly compact. Let $\mathcal{W} = \{W_\alpha \mid \alpha \in A\}$ be a pre-open cover of $X$. By Theorem 2.2, for each finite subcollection $\mathcal{V}$ of $\mathcal{W}$, we have $\bigcup_{V \in \mathcal{V}} \text{Cl}(\text{Int}(V)) \neq X$ which implies that $\bigcap_{V \in \mathcal{V}} \text{Cl}(\text{Int}(X - V)) \neq \emptyset$. So we find that $\mathcal{F} = \{X - W_\alpha \mid \alpha \in A\}$ is a collection of pre-closed sets such that $\bigcap \{\text{Cl}(\text{Int}(E)) \mid E \in \mathcal{E}\} \neq \emptyset$ for each finite subcollection $\mathcal{E}$ of $\mathcal{F}$. By (c), we conclude that $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$. But according to
our assumption \( \bigcup_{\alpha \in A} W_\alpha = X \) which means that \( \bigcap_{\alpha \in A} (X - W_\alpha) = \emptyset \), i.e., 
\( \bigcap \{ F \mid F \in \mathcal{F} \} = \emptyset \), a contradiction. So \( X \) is nearly compact. \( \square \)

**Lemma 2.2** (Mashhour et al. [10]). Let \( A \) and \( B \) be subsets of a topological space \( X \).

(i) If \( A \in \text{PO}(X) \) and \( B \in \text{SO}(X) \), then \( A \cap B \in \text{PO}(B) \).

(ii) If \( A \in \text{PO}(B) \) and \( B \in \text{PO}(X) \), then \( A \in \text{PO}(X) \).

**Theorem 2.5.** If \( A \) is both open and closed in \((X, \mathcal{P})\), then \( A \) is nearly compact with respect to \((A, \mathcal{P}_A)\) if and only if \( A \) is nearly compact with respect to \((X, \mathcal{P})\).

**Proof.** Firstly, suppose that \( A \) is nearly compact with respect to \((X, \mathcal{P})\). Let \( \mathcal{F}^{(A)} \) be a pre-open cover of \( A \) with respect to \((A, \mathcal{P}_A)\). Since \( A \in \text{PO}(X) \), we have \( S \in \text{PO}(X) \) for all \( S \in \mathcal{F}^{(A)} \) by Theorem 2.2. So \( \mathcal{F}^{(X)} = \mathcal{F}^{(A)} \cup \{ X - A \} \) is a pre-open cover of \( X \). By near compactness of \( X \), there exists a finite open super-cover \( \mathcal{G}^{(X)} \) of \( \mathcal{F}^{(X)} \). Since \( \mathcal{G}^{(X)} \) is a cover of \( X \) and \( Cl(X - A) = X - A \), there exists a \( V \in \mathcal{G}^{(X)} \) such that \( V = X - A \) and no \( V \in \mathcal{G}^{(X)} \) with \( X - A \subseteq V \not\subseteq Cl(X - A) = X - A \). Let \( \mathcal{G}^{(A)} \) be obtained from \( \mathcal{G}^{(X)} \) by removing all \( V = X - A \in \mathcal{G}^{(X)} \). It means that for each \( G \in \mathcal{G}^{(A)} \), there exists an \( S \in \mathcal{F}^{(A)} \) such that \( S \subset G \subset Cl_X(S) \). Since \( S \in \mathcal{F}^{(A)} \) are subsets of \( A \), we get \( S = A \cap S \subset A \cap G \subset A \cap Cl_X(S) = Cl_A(S) \). As \( A \cap G \) is open in \((A, \mathcal{P}_A)\), the collection \( \{ A \cap G \mid G \in \mathcal{G}^{(A)} \} \) is a finite open super-cover of \( \mathcal{F}^{(A)} \). So \((A, \mathcal{P}_A)\) is a nearly compact subspace of \((X, \mathcal{P})\).

Conversely, let \( A \) be nearly compact with respect to \((A, \mathcal{P}_A)\) and let \( \mathcal{F}^{(X)} \) be a pre-open cover of \( A \) with respect to \((X, \mathcal{P})\). We have nothing to prove if there exists an \( S \in \mathcal{F}^{(X)} \) such that \( A \subset S \). So we suppose that \( S \not\subset A \) for each \( S \in \mathcal{F}^{(X)} \). Since \( A \in \text{SO}(X) \), we have \( S = A \cap S \in \text{PO}(A) \) for each \( S \in \mathcal{F}^{(X)} \) by Theorem 2.2. So \( \{ S \mid S \in \mathcal{F}^{(X)} \} \) is a pre-open cover of \( A \) with respect to \((A, \mathcal{P}_A)\). By near compactness of \( A \), we obtain a finite open super-cover \( \mathcal{G}^{(A)} \) with respect to \((A, \mathcal{P}_A)\) of \( \mathcal{F}^{(X)} \). For each \( G \in \mathcal{G}^{(A)} \), we have \( S \subset G \subset Cl_A(S) = A \cap Cl_X(S) \subset Cl_X(S) \) for some \( S \in \mathcal{F}^{(X)} \). Thus \( \mathcal{G}^{(A)} \) is a finite open super-cover of \( \mathcal{F}^{(X)} \) with respect to \((X, \mathcal{P})\). \( \square \)

**Definition 2.8.** A topological space \( X \) is said to be pre-regular if for each \( x \in X \) and each closed set \( F \) with \( x \notin F \), there exist a pre-open set \( G \) and an open set \( H \) such that \( x \in G, F \subset H, \) and \( G \cap H = \emptyset \).

It is easy to show that a topological space \( X \) is pre-regular if and only if, for each \( x \) and each open set \( U \) with \( x \in U \), there exists a pre-open set \( V \) such that \( x \in V \subset Cl(V) \subset U \).

**Theorem 2.6.** A pre-regular nearly compact space is a compact space.

**Proof.** Let \( X \) be a pre-regular nearly compact space and let \( \mathcal{G} = \{ G_\alpha \mid \alpha \in A \} \) be an open cover of \( X \). For each \( x \in X \), there exists a \( G_{\alpha(x)} \), \( \alpha(x) \in A \)
such that \( x \in G_{\alpha(x)} \). By pre-regularity of \( X \), we obtain a pre-open set \( V_\alpha(x) \) such that \( x \in V_\alpha(x) \subset \text{Cl}(V_\alpha(x)) \subset G_{\alpha(x)} \). So \( \mathcal{V} = \{ V_\alpha(x) \mid x \in X \} \) is a pre-open cover of \( X \). By near compactness of \( X \), we get a finite open super-cover \( \{ U(x_1), U(x_2), \ldots, U(x_n) \} \) of \( \mathcal{V} \). For each \( k \in \{1, 2, \ldots, n\} \), there exists a \( V_\alpha(x_k) \in \mathcal{V} \) such that \( V_\alpha(x_k) \subset U(x_k) \subset \text{Cl}(V_\alpha(x_k)) \subset G_{\alpha(x_k)} \). Since \( \bigcup_{k=1}^n U(x_k) = X \), the collection \( \{ G_{\alpha(x_1)}, G_{\alpha(x_2)}, \ldots, G_{\alpha(x_n)} \} \) is a finite subcover of \( \mathcal{V} \). \( \square \)

**Theorem 2.7.** If \( E \) is pre-closed and open, and \( F \) is closed such that \( E \cap F = \emptyset \) in a pre-regular nearly compact topological space \( X \), then there exist open sets \( G, H \) in \( X \) such that \( E \subset G, \; F \subset H \), and \( G \cap H = \emptyset \).

**Proof.** For each \( x \in E \), we obtain a pre-open set \( A_x \) and an open set \( H_x \) such that \( x \in A_x, \; F \subset H_x \) and \( A_x \cap H_x = \emptyset \) by pre-regularity of \( X \). This means that \( \mathcal{J} = \{ A_x \mid x \in E \} \cup \{ X - E \} \) is a pre-open cover of \( X \). By near compactness of \( X \), we get a finite open super-cover \( \mathcal{J} \) of \( \mathcal{J} \). We now extract a finite subcollection \( \mathcal{J}(E) = \{ G_1, G_2, \ldots, G_n \} \) from \( \mathcal{J} \) to cover \( E \), and \( \mathcal{J}(E) \) is associated to \( A_x \mid x \in E \). For each \( k \in \{1, 2, \ldots, n\} \), we obtain \( A_{x_k} \subset A_x \mid x \in E \) such that \( A_{x_k} \subset G_k \subset \text{Cl}(A_{x_k}) \). We write \( G = \bigcup_{k=1}^n G_k \) and \( H = \bigcap_{k=1}^n H_{x_k} \). We see that \( E \subset G, \; F \subset H \). Now we show that \( G \cap H = \emptyset \). Suppose for contradiction that \( G \cap H \neq \emptyset \) and let \( z \in G \cap H \). So \( z \in G_k \) for some \( k \in \{1, 2, \ldots, n\} \) and \( z \in H_{x_k} \) for each \( k \in \{1, 2, \ldots, n\} \). Let \( z \in G_l \) for some \( l \in \{1, 2, \ldots, n\} \). So \( z \in \text{Cl}(A_{x_l}) \). This implies that \( \text{Cl}(A_{x_l}) \cap H_{x_l} \neq \emptyset \), which is a contradiction to the fact that \( \text{Cl}(A_{x_l}) \cap H_{x_l} = \emptyset \). \( \square \)

Recall that a topological space \( X \) is called extremally disconnected if the closure of each open set in \( X \) is open. Sometimes Hausdorffness is also included in the definition of extremal disconnectedness of a topological space.

**Theorem 2.8.** A quasi \( H \)-closed extremally disconnected topological space is nearly compact.

**Proof.** Let \( \mathcal{A} = \{ A_\alpha \mid \alpha \in \Delta \} \) be a pre-open cover of a quasi \( H \)-closed extremally disconnected topological space \( X \). For each \( \alpha \in \Delta \), there exists an open set \( G_\alpha \) such that \( A_\alpha \subset G_\alpha \subset \text{Cl}(A_\alpha) = \text{Cl}(G_\alpha) \). We see that \( \mathcal{A} = \{ G_\alpha \mid \alpha \in \Delta \} \) is an open cover of \( X \). By quasi \( H \)-closedness of \( X \), we get a finite subcollection \( \{ G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n} \} \) such that \( \{ \text{Cl}(G_{\alpha_k}) \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\} \} \) covers \( X \). It now follows by extremal disconnectedness of \( X \) that \( \{ \text{Cl}(G_{\alpha_k}) \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\} \} \) is a finite open super-cover of \( \mathcal{A} \). \( \square \)

**Definition 2.9** (Mukharjee et al. [12]). A semi-open set \( A \) in \( X \) is called covered if whenever \( G \subset A \subset \text{Cl}(G) \) for some open set \( G \), there exists an open set \( H \) such that \( G \subset A \subset H \subset \text{Cl}(G) \).
Lemma 2.3 (Mukharjee et al. [12]). A covered semi-open set in $X$ is pre-open in $X$.

Theorem 2.9. If each semi-open subset of a nearly compact space $X$ is covered, then $X$ is $S$-closed.

Proof. Let $\mathcal{I}$ be a semi-open cover of $X$. By Lemma 2.3, $\mathcal{I}$ is a pre-open cover of $X$. By Theorem 2.2, $\mathcal{I}$ has a finite subcollection $\mathcal{I}$ such that $\{\text{Int}(\text{Cl}(A)) \mid A \in \mathcal{I}\}$ covers $X$. For each $A \in \mathcal{I}$, we have $A \subset \text{Int}(\text{Cl}(A)) \subset \text{Cl}(A)$. So $\mathcal{I}$ is a finite subcollection of $\mathcal{I}$ such that $\{\text{Int}(\text{Cl}(A)) \mid A \in \mathcal{I}\}$ covers $X$, and so $X$ is $S$-closed. □

A subset $A$ of $X$ is said to be nearly compact with respect to $X$ if each pre-open cover with respect to $X$ of $A$ has a finite open super-cover. In view of Theorem 2.2, it can be showed that a subset $A$ of $X$ is nearly compact with respect to $X$ if each pre-open cover $\mathcal{I}$ with respect to $X$ of $A$ has a finite subcollection $T$ such that $\{\text{Int}(\text{Cl}(G)) \mid G \in T\}$ covers $A$.

Theorem 2.10. If each proper regularly closed set of $X$ is nearly compact with respect to $X$, then $X$ is nearly compact.

Proof. Let $\mathcal{I} = \{A_\alpha \mid \alpha \in \Delta\}$ be a pre-open cover of $X$. Since $\mathcal{I}$ is a cover of $X$, there exits an $A \in \mathcal{I}$ such that $A \neq \emptyset$. By Lemma 2.1, $\text{Int}(\text{Cl}(A))$ is regularly open in $X$, and so $X - \text{Int}(\text{Cl}(A))$ is regularly closed in $X$. By the assumption, we get a finite subcollection $\{A_{\alpha_k} \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\}$ such that

$$X - \text{Int}(\text{Cl}(A)) \subset \bigcup_{k=1}^n \text{Int}(\text{Cl}(A_{\alpha_k}))$$

and thus

$$X \subset \bigcup_{k=1}^n (\text{Int}(\text{Cl}(A_{\alpha_k}))) \cup \text{Int}(\text{Cl}(A)).$$

Therefore, by Theorem 2.2, $X$ is nearly compact. □

Recall that a nonempty collection $\mathcal{F}$ of nonempty subsets of a set $X$ is called a filter base (see [13, p. 49]) if whenever $F_1, F_2 \in \mathcal{F}$, one has $F_3 \subset F_1 \cap F_2$ for some $F_3 \in \mathcal{F}$. A filter base is called maximal (see [13, p. 47]) if it is not properly contained in another filter base. A filter base is always contained in a maximal filter base (see [13, p. 47]).

Definition 2.10. A filter base $\mathcal{F}$ on $X$ is said to $p$-converge to a point $x \in X$ if, for each pre-open set $A$ of $X$, with $x \in A$, there exists $F \in \mathcal{F}$ such that $F \subset \text{Int}(\text{Cl}(A))$.

Definition 2.11. A filter base $\mathcal{F}$ in $X$ is said to $p$-accumulate to a point $x \in X$ if, for each pre-open set $A$ of $X$ with $x \in A$, one has $F \cap (\text{Int}(\text{Cl}(A))) \neq \emptyset$ for each $F \in \mathcal{F}$.
The following lemmas 2.4, 2.5, and 2.6 are easy to establish and hence their proofs are omitted.

**Lemma 2.4.** If a filter base $\mathcal{F}$ in $X$ $p$-converges to a point $x \in X$, then the filter base $p$-accumulates to $x$.

**Lemma 2.5.** Let $\mathcal{F}$ be a maximal filter base in $X$. Then $\mathcal{F}$ $p$-converges to $x \in X$ if and only if $\mathcal{F}$ $p$-accumulates to $x \in X$.

**Lemma 2.6.** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filter bases in $X$ such that $\mathcal{F}_2$ is a subcollection of $\mathcal{F}_1$. Then $\mathcal{F}_2$ $p$-accumulates to a point $x \in X$ if $\mathcal{F}_1$ $p$-accumulates to $x \in X$.

**Theorem 2.11.** The following statements are equivalent.

(a) $X$ is nearly compact.

(b) Each maximal filter base $p$-converges in $X$.

(c) Each filter base $p$-accumulates to some $x_0 \in X$.

(d) For each family $\mathcal{E}$ of pre-closed sets with $\bigcap_{F \in \mathcal{E}} F = \emptyset$, there exists a finite subcollection $\Delta$ of $\mathcal{E}$ such that $\bigcap_{E \in \Delta} \text{Cl}(\text{Int}(E)) = \emptyset$.

**Proof.** (a)$\Rightarrow$(b). Let $\mathcal{F} = \{A_\alpha \mid \alpha \in \Delta\}$ be a maximal filter base in $X$. Suppose for contradiction that $\mathcal{F}$ does not $p$-converge to a point of $X$. By Lemma 2.5, $\mathcal{F}$ does also not $p$-accumulate to a point of $X$. This means that, for each $x \in X$, there exist a pre-open set $G_x$ containing $x$ and $A_{\alpha(x)}, \alpha(x) \in \Delta$ such that $(\text{Int}(\text{Cl}(G_x))) \cap A_{\alpha(x)} = \emptyset$. Then $\mathcal{F} = \{G_x \mid x \in X\}$ is a pre-open cover of $X$. By Theorem 2.2, we obtain a finite subcollection $\{G_{x_1}, G_{x_2}, \ldots, G_{x_n}\}$ of $\mathcal{F}$ such that $\bigcup_{k=1}^{n} \text{Int}(\text{Cl}(G_{x_k})) = X$.

Since $\mathcal{F}$ is a filter base, there exists a $A_0 \in \mathcal{F}$ such that $A_0 \subset \bigcap_{k=1}^{n} A_{x_k}$. So $(\text{Int}(\text{Cl}(G_{x_k}))) \cap A_0 = \emptyset$ for each $k \in \{1, 2, \ldots, n\}$. We see that

\[
A_0 = X \cap A_0 = \left(\bigcup_{k=1}^{n} \text{Int}(\text{Cl}(G_{x_k}))\right) \cap A_0 = \bigcup_{k=1}^{n} (\text{Int}(\text{Cl}(G_{x_k})) \cap A_0) = \emptyset,
\]

a contradiction to the fact that $A_0 \neq \emptyset$.

(b)$\Rightarrow$(c). Let $\mathcal{F}$ be a filter base in $X$. Then there exists a maximal filter base $\mathcal{E}$ containing $\mathcal{F}$ as a subcollection. By (b), the filter base $\mathcal{E}$ $p$-converges to some $x_0 \in X$. By Lemma 2.4, $\mathcal{E}$ $p$-accumulates to $x_0$ and thus, by Lemma 2.6, $\mathcal{F}$ $p$-accumulates to $x_0$.

(c)$\Rightarrow$(d). Let $\mathcal{E} = \{A_\alpha \mid \alpha \in \Delta\}$ be a collection of pre-closed subsets of $X$ such that $\bigcap_{\alpha \in \Delta} A_\alpha = \emptyset$. Suppose for contradiction that for each finite subcollection $\Delta_0$ of $\Delta$, we have $\bigcap_{\alpha \in \Delta_0} \text{Cl}(\text{Int}(A_\alpha)) \neq \emptyset$. Set $F_{\Delta_0} = \bigcap_{\alpha \in \Delta_0} \text{Cl}(\text{Int}(A_\alpha))$. Let $\Lambda$ be the collection of all finite subcollection of $\Delta$. We write $\mathcal{F} = \{F_\lambda \mid \lambda \in \Lambda\}$ (each $F_\lambda$ bears the meaning as of $F_{\Delta_0}$). We see that $\mathcal{F}$ is a filter base in $X$. By (c), $\mathcal{F}$ $p$-accumulates to some point.
\[ x_0 \in X. \] This means that \( F_\lambda \cap (\text{Cl}(\text{Int}(A))) \neq \emptyset \) for each \( \lambda \in \Lambda \) and each pre-open set \( A \) of \( X \) containing \( x_0 \), in particular, 
\[ \text{Cl}(\text{Int}(A_\alpha)) \cap A \neq \emptyset \tag{2.1} \]
for each \( \alpha \in \Delta \) and each pre-open set \( A \) containing \( x_0 \). As \( \bigcap_{\alpha \in \Delta} A_\alpha = \emptyset \), we have \( x_0 \notin \bigcap_{\alpha \in \Delta} A_\alpha \) and thus \( x_0 \notin A_{\alpha_0} \) for some \( \alpha_0 \in \Delta \). So we get a pre-open set \( X - A_{\alpha_0} \) such that that \( x_0 \in X - A_{\alpha_0} \). According to the construction, \( \text{Cl}(\text{Int}(A_{\alpha_0})) \in \mathcal{F} \). Now 
\[ (\text{Int}(\text{Cl}(X - A_{\alpha_0})) \cap (\text{Cl}(\text{Int}(A_{\alpha_0}))) = (X - \text{Cl}(\text{Int}(A_{\alpha_0}))) \cap (\text{Cl}(\text{Int}(A_{\alpha_0}))) = \emptyset, \]
a contradiction to (2.1).

(d)\(\Rightarrow\)(a). Suppose that \( \mathcal{S} = \{A_\alpha \mid \alpha \in \Delta\} \) is a pre-open cover of \( X \). Then \( \{X - A_\alpha \mid \alpha \in \Delta\} \) is a collection of pre-closed sets such that \( \bigcap_{\alpha \in \Delta}(X - A_\alpha) = \emptyset \). By (d), we obtain a finite subcollection \( \Delta_0 \) of \( \Delta \) such that 
\[ \bigcap_{\alpha \in \Delta_0}\text{Cl}(\text{Int}(X - A_\alpha)) = \emptyset, \] which in turn implies that \( \bigcup_{\alpha \in \Delta_0}\text{Int}(\text{Cl}(A_\alpha)) = X \). So, by Theorem 2.2, \( X \) is nearly compact. \( \square \)

Remark 1. A subset \( A \) of \( X \) is \( \alpha \)-set (see [14]) if \( A \subset \text{Int}(\text{Cl}(\text{Int}(A))) \). So a subset \( A \) of \( X \) is \( \alpha \)-set if and only if it is an open cover of \( X \). Then 
\[ \{X - A_\alpha \mid \alpha \in \Delta\} \] is a collection of closed sets such that \( \bigcap_{\alpha \in \Delta}(X - A_\alpha) = \emptyset \). By (d), we obtain a finite subcollection \( \Delta_0 \) of \( \Delta \) such that 
\[ \bigcap_{\alpha \in \Delta_0}\text{Cl}(\text{Int}(X - A_\alpha)) = \emptyset, \] which in turn implies that \( \bigcup_{\alpha \in \Delta_0}\text{Int}(\text{Cl}(A_\alpha)) = X \). So, by Theorem 2.2, \( X \) is nearly compact.

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References

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