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A new approach to nearly compact spaces

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Abstract. Using the covers formed by pre-open sets, we introduce and study the notion of po-compactness in topological spaces. The notion of po-compactness is weaker than that of compactness but stronger than semi-compactness. It is observed that po-compact spaces are the same as nearly compact spaces. However, we find new characterizations to near compactness, when we study it in the sense of po-compactness.

1. Introduction

Unless otherwise mentioned, X stands for the topological space $(X, \mathscr{P})$. $Int_X(A)$ or $Int(A)$ (respectively, $Cl_X(A)$ or $Cl(A)$) denotes the interior (respectively, closure) of a subset A in a topological space X.

Generalizing the concept of open sets, Levine [7] introduced the notion of semi-open sets: a subset A of a topological space X is semi-open if there exists an open set G such that $G \subset A \subset Cl(G)$. One more generalization of open sets is the notion of pre-open sets (see Mashhour et al. [8]) introduced by Corson and Michael [3] under the name locally dense sets: a subset A of a topological space X is called locally dense if there exists an open set U such that $A \subset U \subset Cl(A)$. The complement of a pre-open set is called a pre-closed set (see [8]). The concepts of semi-open sets and pre-open sets are independent. Several covering properties have been introduced and studied using covers formed by semi-open and pre-open sets (see, for example, [2, 6, 9, 8, 11, 16, 17]). A cover formed by semi-open sets is called an s-cover (see [16]). A topological space X is called s-compact (see [16]) if each s-cover of X has a finite subcover. However, s-compactness has been widely studied under the name semi-compactness by Dorsett [4]. As semi-compactness is

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stronger than compactness, and the term semi-compact delineates a notion weaker than compactness, we retain the term s-compactness due to Prasad and Yadav [16] to mean the notion of semi-compactness due to Dorsett [4] as well, and henceforth, by semi-compactness, we mean the notion introduced by Mukharjee et al. [11]. Mashhour et al. [9] introduced a compact like notion called strong compactness: a topological space X is called strongly compact if each cover of X by pre-open sets of X has a finite subcover.

For a topological space $(X, \mathscr{P})$ and a subset $A \subset X$, we write $(A, \mathscr{P}_A)$ to denote the subspace on A of $(X, \mathscr{P})$. We also write $SO(X)$ (respectively, $PO(X)$) to denote the collection of all semi-open (respectively, pre-open) sets of X. Throughout the paper, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers.

2. Po-compactness

We begin by recalling that a subset A of a topological space X is called regularly open if $A = Int(Cl(A))$. So if G is open in X, then $Int(Cl(G))$ is regularly open in X.

We agree to mean by an open collection and pre-open collection a collection consisting, respectively, of open sets and pre-open sets of a topological space X. An open (respectively, a pre-open) collection $\mathscr{A}$ of subsets of X such that $\bigcup_{A \in \mathscr{A}} A = X$ is called an open cover (respectively, a pre-open cover) of X. The terms regularly open collection and regularly open cover are apparent. A collection $\mathscr{A}$ of subsets of X is called a weak cover of X if $Cl\left(\bigcup_{A \in \mathscr{A}} A\right) = X$.

Definition 2.1 (Singal and Mathur [18])**.** A topological space X is called nearly compact if, for each open cover $\mathscr{U}$ of X, there exists a finite subcollection $\mathscr{V} \subset \mathscr{U}$ such that $\bigcup\{Int(Cl(V)) \mid V \in \mathscr{V}\} = X$.

Definition 2.2 (Alexandroff and Urysohn [1], see also [5] or [15])**.** A Hausdorff topological space is called H-closed if the space is closed in every Hausdorff topological space containing it as a subspace.

It is seen that a Hausdorff topological space X is H-closed if and only if each open cover $\mathscr{U}$ of X has a finite subcollection $\mathscr{V} \subset \mathscr{U}$ such that $\bigcup\{Cl(V) \mid V \in \mathscr{V}\} = X$.

Dropping the Hausdorffness from the notion of H-closedness, we get a type of covering notion called quasi H-closed spaces (see [15]).

Definition 2.3 (Thompson [19])**.** A topological space X is called S-closed if for each s-cover $\mathscr{U}$ of X, there exists a finite subcollection $\mathscr{V} \subset \mathscr{U}$ such that $\{Cl(V) \mid V \in \mathscr{V}\}$ covers X.

Definition 2.4 (Mukharjee et al. [11]). Let $\mathscr{U}$ be a collection of open sets of X. Then the collection

$$\mathscr{V} = \{V \mid U \in \mathscr{U}, U \subset V \subset Cl(U), V \neq Cl(U) \text{ when } Cl(U) \notin \mathscr{P}\}$$

is called a semi-open super-collection of $\mathscr{U}$.

We note that $\mathscr{V}$ is a cover of X if $\mathscr{U}$ is a cover of X. In this case, $\mathscr{V}$ is called a semi-open super-cover of the open cover $\mathscr{U}$.

Definition 2.5 (Mukharjee et al. [11]). A topological space X is called semi-compact if each open cover of X has a finite semi-open super-cover.

We now introduce the following definitions.

Definition 2.6. Let $\mathscr{S}$ be a pre-open collection of X. If, for each $A \in \mathscr{S}$, there exists an open set U such that $A \subset U \subset Cl(A)$, then the collection

$$\mathscr{U} = \{U \mid A \in \mathscr{S}, A \subset U \subset Cl(A)\}$$

is said to be an open super-collection of $\mathscr{S}$.

Note that there always exists an open super-collection of a pre-open collection of a topological space X. Let us also note that $\mathscr{U}$ is a cover of X if $\mathscr{S}$ is a cover of X. In this case, $\mathscr{U}$ is said to be an open super-cover of the pre-open cover $\mathscr{S}$.

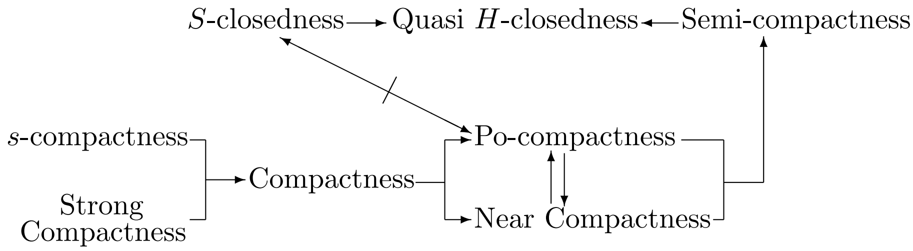
Definition 2.7. A topological space X is said to be po-compact if each pre-open cover of X has a finite open super-cover.

Let $\mathscr{U}$ be a finite open super-cover of a pre-open cover $\mathscr{S}$ of a po-compact space X. For each $U \in \mathscr{U}$, there exists a pre-open set $A \in \mathscr{S}$ such that $A \subset U \subset Cl(A)$. Thus we have a finite subcollection $\{A \mid U \in \mathscr{U}, A \subset U \subset Cl(A)\}$ of $\mathscr{S}$ corresponding to $\mathscr{U}$.

It is easy to see that compact and hence strongly compact spaces, and s-compact spaces are po-compact spaces. However, a po-compact space need not be a compact space, and a semi-compact space need not be a po-compact space.

Example 1. Let $\mathscr{P} = \{(-\infty, n) \mid n \in \mathbb{N}\}$. The topological space $(\mathbb{R}, \mathscr{P})$ is po-compact, but not compact.

We summarize the implication relations of po-compactness with compactness and compact like notions in the following diagram for better understanding its position.



In the figure, "$P \to Q$" stands to mean "P implies Q" and "$P \not\to Q$" stands to mean "P does not imply Q".

Theorem 2.1. *A space X is po-compact if and only if it is nearly compact.*

Proof. The necessity follows directly from the fact that open sets in a topological space X are pre-open.

Conversely, let X be a nearly compact space and let $\mathscr{S}$ be a pre-open cover of X. For each $A \in \mathscr{S}$, there exists an open set G such that $A \subset G \subset Cl(A)$. Since $\mathscr{S}$ is a cover of X, the collection $\mathscr{G} = \{G \in \mathscr{P} \mid A \subset G \subset Cl(A), A \in \mathscr{S}\}$ is an open cover of X. By near compactness of X, the collection $\mathscr{G}$ has a finite subcollection $\mathscr{G}_n = \{G_k \in \mathscr{G} \mid k \in \{1, 2, \dots, n\}\}$ such that $\bigcup_{k=1}^{n} Int(Cl(G_k)) = X$. For each $k \in \{1, 2, \dots, n\}$, there exists an $A_k \in \mathscr{S}$ such that $A_k \subset G_k \subset Cl(A_k)$. It means that $Cl(A_k) = Cl(G_k)$ which implies that $A_k \subset Int(Cl(G_k)) = Int(Cl(A_k)) \subset Cl(A_k)$ for each $k \in \{1, 2, \dots, n\}$. So $\{Int(Cl(G_k)) \mid G_k \in \mathscr{G}_n, k \in \{1, 2, \dots, n\}\}$ is a finite open super-cover of $\mathscr{S}$. $\square$

Lemma 2.1. *If A is pre-open, then $Int(Cl(A))$ is regularly open.*

Proof. If A is a pre-open set, then there exists an open set G such that $Cl(A) = Cl(G)$. So $Int(Cl(A))$ is regularly open. $\square$

Theorem 2.2. *A topological space X is nearly compact if and only if each pre-open cover $\mathscr{A}$ of X has a finite regularly open super-cover $\{Int(Cl(A)) \mid A \in \mathscr{B}\}$, where $\mathscr{B}$ is a finite subcollection of $\mathscr{A}$.*

Proof. Suppose that $\mathscr{A}$ is a pre-open cover of a nearly compact topological space X. By near compactness of X, the cover $\mathscr{A}$ has a finite open super-cover $\mathscr{G}$. For each $G \in \mathscr{G}$, we get a pre-open set $A \in \mathscr{A}$ such that $A \subset G \subset Cl(A)$ and hence $A \subset G \subset Int(Cl(A)) \subset Cl(A)$. Thus we get a finite subcollection $\mathscr{B} = \{A \in \mathscr{A} \mid G \in \mathscr{G}, A \subset G \subset Cl(A)\}$ of $\mathscr{A}$. As $\mathscr{G}$ is a cover of X, $\{Int(Cl(A)) \mid A \in \mathscr{B}\}$ is also a cover of X. By Lemma 2.1, $Int(Cl(B))$ is regularly open for each $B \in \mathscr{B}$. So $\mathscr{B}$ is a finite subcollection of $\mathscr{A}$ such that $\{Int(Cl(B)) \mid B \in \mathscr{B}\}$ is a regularly open super-cover of the pre-open cover $\mathscr{A}$ of X.

Conversely, since $Int(Cl(A))$ is open and $A \subset Int(Cl(A)) \subset Cl(A)$ for each $A \in \mathscr{B}$, the collection $\{Int(Cl(A)) \mid A \in \mathscr{B}\}$ is a finite open super-cover of $\mathscr{A}$. So X is nearly compact. □

Corollary 2.1. *If $\mathscr{F}$ is a collection of pre-closed sets such that $\bigcap_{F\in\mathscr{F}} F = \emptyset$ in a nearly compact space X, then there exists a finite subcollection $\mathscr{E}$ of $\mathscr{F}$ such that $\bigcap_{E\in\mathscr{E}} Cl(Int(E)) = \emptyset$.*

Proof. Since $\mathscr{F}$ is a collection of pre-closed sets satisfying $\bigcap_{F\in\mathscr{F}} F = \emptyset$, the collection $\{X - F \mid F \in \mathscr{F}\}$ is a pre-open cover of X. By Theorem 2.2, we get a finite subcollection $\{X - F_1, X - F_2, \ldots, X - F_n\}$ of $\mathscr{F}$ such that $\bigcup_{k=1}^{n} Int(Cl(X - F_k)) = X$. Hence $\bigcap_{k=1}^{n} Cl(Int(F_k)) = \emptyset$. □

Theorem 2.3. *Each pre-open cover of a nearly compact space has a finite pre-open weak cover.*

Proof. Let X be a nearly compact space and let $\mathscr{S}$ be a pre-open cover of X. By near compactness of X, the cover $\mathscr{S}$ has a finite open super-cover $\mathscr{G}$. For each $G \in \mathscr{G}$, there exists an $A \in \mathscr{S}$ such that $A \subset G \subset Cl(A)$ which implies that $Cl(A) = Cl(G)$. So it follows that $\mathscr{H} = \{A \mid A \subset G \subset Cl(A), G \in \mathscr{G}\}$ is a finite pre-open weak cover of X. □

Theorem 2.4. *In a topological space X, the following statements are equivalent.*

(a) *X is nearly compact.*
(b) *Each pre-open cover $\mathscr{A}$ of X has a finite subcollection $\mathscr{B}$ such that $\{Int(Cl(B)) \mid B \in \mathscr{B}\}$ covers X.*
(c) *Each family $\mathscr{F}$ of pre-closed sets has nonempty intersection if $\bigcap\{(Cl(Int(E)) \mid E \in \mathscr{E}\} \neq \emptyset$ for each finite subcollection $\mathscr{E}$ of $\mathscr{F}$.*

Proof. (a)⇒(b) follows from Theorem 2.2.

(b)⇒(c). Let $\mathscr{F} = \{F_\alpha \mid \alpha \in A\}$ be a collection of pre-closed sets of X such that for each finite subfamily $\mathscr{E}$ of $\mathscr{F}$, $\bigcap\{(Cl(Int(E)) \mid E \in \mathscr{E}\} \neq \emptyset$. If possible, let $\bigcap_{F\in\mathscr{F}} F = \emptyset$. Then $\mathscr{G} = \{X - F_\alpha \mid \alpha \in A\}$ is a pre-open cover of X. By (b), we get a finite subcollection $\{X - F_{\alpha_k} \mid k \in \{1, 2, \ldots, n\}\}$ of $\mathscr{G}$ such that $\{Int(Cl(X - F_{\alpha_k})) \mid k \in \{1, 2, \ldots, n\}\}$ covers X. It means that $X - \bigcup_{k=1}^{n} Int(Cl(X - F_{\alpha_k})) = \emptyset$ which in turn implies that $\bigcap_{k=1}^{n} Cl(Int(F_{\alpha_k})) = \emptyset$, a contradiction to our assumption.

(c)⇒(a). Let X be a topological space satisfying (c). Suppose for contradiction that X is not nearly compact. Let $\mathscr{W} = \{W_\alpha \mid \alpha \in A\}$ be a pre-open cover of X. By Theorem 2.2, for each finite subcollection $\mathscr{V}$ of $\mathscr{W}$, we have $\bigcup_{V\in\mathscr{V}} Int(Cl(V)) \neq X$ which implies that $\bigcap_{V\in\mathscr{V}} Cl(Int(X - V)) \neq \emptyset$. So we find that $\mathscr{F} = \{X - W_\alpha \mid \alpha \in A\}$ is a collection of pre-closed sets such that $\bigcap\{(Cl(Int(E)) \mid E \in \mathscr{E}\} \neq \emptyset$ for each finite subcollection $\mathscr{E}$ of $\mathscr{F}$. By (c), we conclude that $\bigcap\{F \mid F \in \mathscr{F}\} \neq \emptyset$. But according to our assumption $\bigcup_{\alpha\in A} W_\alpha = X$ which means that $\bigcap_{\alpha\in A}(X - W_\alpha) = \emptyset$, i.e., $\bigcap\{F \mid F \in \mathscr{F}\} = \emptyset$, a contradiction. So X is nearly compact. $\square$

Lemma 2.2 (Mashhour et al. [10]). *Let A and B be subsets of a topological space X.*

(i) *If $A \in PO(X)$ and $B \in SO(X)$, then $A \cap B \in PO(B)$.*
(ii) *If $A \in PO(B)$ and $B \in PO(X)$, then $A \in PO(X)$.*

Theorem 2.5. *If A is both open and closed in $(X, \mathscr{P})$, then A is nearly compact with respect to $(A, \mathscr{P}_A)$ if and only if A is nearly compact with respect to $(X, \mathscr{P})$.*

Proof. Firstly, suppose that A is nearly compact with respect to $(X, \mathscr{P})$. Let $\mathscr{S}^{(A)}$ be a pre-open cover of A with respect to $(A, \mathscr{P}_A)$. Since $A \in PO(X)$, we have $S \in PO(X)$ for all $S \in \mathscr{S}^{(A)}$ by Theorem 2.2. So $\mathscr{S}^{(X)} = \mathscr{S}^{(A)} \cup \{X - A\}$ is a pre-open cover of X. By near compactness of X, there exists a finite open super-cover $\mathscr{G}^{(X)}$ of $\mathscr{S}^{(X)}$. Since $\mathscr{G}^{(X)}$ is a cover of X and $Cl(X - A) = X - A$, there exists a $V \in \mathscr{G}^{(X)}$ such that $V = X - A$ and no $V \in \mathscr{G}^{(X)}$ with $X - A \subsetneqq V \subsetneqq Cl(X - A) = X - A$. Let $\mathscr{G}^{(A)}$ be obtained from $\mathscr{G}^{(X)}$ by removing all $V = X - A \in \mathscr{G}^{(X)}$. It means that for each $G \in \mathscr{G}^{(A)}$, there exists an $S \in \mathscr{S}^{(A)}$ such that $S \subset G \subset Cl_X(S)$. Since $S \in \mathscr{S}^{(A)}$ are subsets of A, we get $S = A \cap S \subset A \cap G \subset A \cap Cl_X(S) = Cl_A(S)$. As $A \cap G$ is open in $(A, \mathscr{P}_A)$, the collection $\{A \cap G \mid G \in \mathscr{G}^{(A)}\}$ is a finite open super-cover of $\mathscr{S}^{(A)}$. So $(A, \mathscr{P}_A)$ is a nearly compact subspace of $(X, \mathscr{P})$.

Conversely, let A be nearly compact with respect to $(A, \mathscr{P}_A)$ and let $\mathscr{S}^{(X)}$ be a pre-open cover of A with respect to $(X, \mathscr{P})$. We have nothing to prove if there exists an $S \in \mathscr{S}^{(X)}$ such that $A \subset S$. So we suppose that $S \subsetneqq A$ for each $S \in \mathscr{S}^{(X)}$. Since $A \in SO(X)$, we have $S = A \cap S \in PO(A)$ for each $S \in \mathscr{S}^{(X)}$ by Theorem 2.2. So $\{S \mid S \in \mathscr{S}^{(X)}\}$ is a pre-open cover of A with respect to $(A, \mathscr{P}_A)$. By near compactness of A, we obtain a finite open super-cover $\mathscr{G}^{(A)}$ with respect to $(A, \mathscr{P}_A)$ of $\mathscr{S}^{(X)}$. For each $G \in \mathscr{G}^{(A)}$, we have $S \subset G \subset Cl_A(S) = A \cap Cl_X(S) \subset Cl_X(S)$ for some $S \in \mathscr{S}^{(X)}$. Thus $\mathscr{G}^{(A)}$ is a finite open super-cover of $\mathscr{S}^{(X)}$ with respect to $(X, \mathscr{P})$. $\square$

Definition 2.8. A topological space X is said to be pre-regular if for each $x \in X$ and each closed set F with $x \notin F$, there exist a pre-open set G and an open set H such that $x \in G$, $F \subset H$, and $G \cap H = \emptyset$.

It is easy to show that a topological space X is pre-regular if and only if, for each x and each open set U with $x \in U$, there exists a pre-open set V such that $x \in V \subset Cl(V) \subset U$.

Theorem 2.6. *A pre-regular nearly compact space is a compact space.*

Proof. Let X be a pre-regular nearly compact space and let $\mathscr{G} = \{G_\alpha \mid \alpha \in A\}$ be an open cover of X. For each $x \in X$, there exists a $G_{\alpha(x)}, \alpha(x) \in A$

such that $x \in G_{\alpha(x)}$. By pre-regularity of X, we obtain a pre-open set $V_{\alpha(x)}$ such that $x \in V_{\alpha(x)} \subset Cl(V_{\alpha(x)}) \subset G_{\alpha(x)}$. So $\mathscr{V} = \{V_{\alpha(x)} \mid x \in X\}$ is a pre-open cover of X. By near compactness of X, we get a finite open super-cover $\{U_{(x_1)}, U_{(x_2)}, \ldots, U_{(x_n)}\}$ of $\mathscr{V}$. For each $k \in \{1, 2, \ldots, n\}$, there exists a $V_{\alpha(x_k)} \in \mathscr{V}$ such that $V_{\alpha(x_k)} \subset U_{(x_k)} \subset Cl(V_{\alpha(x_k)}) \subset G_{\alpha(x_k)}$. Since $\bigcup_{k=1}^{n} U_{(x_k)} = X$, the collection $\{G_{\alpha(x_1)}, G_{\alpha(x_2)}, \ldots, G_{\alpha(x_n)}\}$ is a finite subcover of $\mathscr{G}$. □

Theorem 2.7. *If E is pre-closed and open, and F is closed such that $E \cap F = \emptyset$ in a pre-regular nearly compact topological space X, then there exist open sets G, H in X such that $E \subset G$, $F \subset H$, and $G \cap H = \emptyset$.*

Proof. For each $x \in E$, we obtain a pre-open set A_x and an open set H_x such that $x \in A_x, F \subset H_x$ and $A_x \cap H_x = \emptyset$ by pre-regularity of X. This means that $\mathscr{S} = \{A_x \mid x \in E\} \cup \{X - E\}$ is a pre-open cover of X. By near compactness of X, we get a finite open super-cover $\mathscr{G}$ of $\mathscr{S}$. We now extract a finite subcollection $\mathscr{G}^{(E)} = \{G_1, G_2, \ldots, G_n\}$ from $\mathscr{G}$ to cover E, and $\mathscr{G}^{(E)}$ is associated to $\{A_x \mid x \in E\}$. For each $k \in \{1, 2, \ldots, n\}$, we obtain $A_{x_k} \in \{A_x \mid x \in E\}$ such that $A_{x_k} \subset G_k \subset Cl(A_{x_k})$. We write $G = \bigcup_{k=1}^{n} G_k$ and $H = \bigcap_{k=1}^{n} H_{x_k}$. We see that $E \subset G, F \subset H$. Now we show that $G \cap H = \emptyset$. Suppose for contradiction that $G \cap H \neq \emptyset$ and let $z \in G \cap H$. So $z \in G_k$ for some $k \in \{1, 2, \ldots, n\}$ and $z \in H_{x_k}$ for each $k \in \{1, 2, \ldots, n\}$. Let $z \in G_l$ for some $l \in \{1, 2, \ldots, n\}$. So $z \in Cl(A_{x_l})$. This implies that $Cl(A_{x_l}) \cap H_{x_l} \neq \emptyset$, which is a contradiction to the fact that $Cl(A_{x_l}) \cap H_{x_l} = \emptyset$. □

Recall that a topological space X is called extremally disconnected if the closure of each open set in X is open. Sometimes Hausdorffness is also included in the definition of extremal disconnectedness of a topological space.

Theorem 2.8. *A quasi H-closed extremally disconnected topological space is nearly compact.*

Proof. Let $\mathscr{A} = \{A_\alpha \mid \alpha \in \Delta\}$ be a pre-open cover of a quasi H-closed extremally disconnected topological space X. For each $\alpha \in \Delta$, there exists an open set G_α such that $A_\alpha \subset G_\alpha \subset Cl(A_\alpha) = Cl(G_\alpha)$. We see that $\mathscr{G} = \{G_\alpha \mid \alpha \in \Delta\}$ is an open cover of X. By quasi H-closedness of X, we get a finite subcollection $\{G_{\alpha_k} \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\}$ such that $\{Cl(G_{\alpha_k}) \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\}$ covers X. It now follows by extremal disconnectedness of X that $\{Cl(G_{\alpha_k}) \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\}$ is a finite open super-cover of $\mathscr{A}$. □

Definition 2.9 (Mukharjee et al. [12])**.** A semi-open set A in X is called covered if whenever $G \subset A \subset Cl(G)$ for some open set G, there exists an open set H such that $G \subset A \subset H \subset Cl(G)$.

Lemma 2.3 (Mukharjee et al. [12]). *A covered semi-open set in X is pre-open in X.*

Theorem 2.9. *If each semi-open subset of a nearly compact space X is covered, then X is S-closed.*

Proof. Let $\mathscr{S}$ be a semi-open cover of X. By Lemma 2.3, $\mathscr{S}$ is a pre-open cover of X. By Theorem 2.2, $\mathscr{S}$ has a finite subcollection $\mathscr{T}$ such that $\{Int(Cl(A)) \mid A \in \mathscr{T}\}$ covers X. For each $A \in \mathscr{T}$, we have $A \subset Int(Cl(A)) \subset Cl(A)$. So $\mathscr{T}$ is a finite subcollection of $\mathscr{S}$ such that $\{(Cl(A) \mid A \in \mathscr{T}\}$ covers X, and so X is S-closed. □

A subset A of X is said to be nearly compact with respect to X if each pre-open cover with respect to X of A has a finite open super-cover. In view of Theorem 2.2, it can be showed that a subset A of X is nearly compact with respect to X if each pre-open cover $\mathscr{S}$ with respect to X of A has a finite subcollection $\mathscr{T}$ such that $\{Int(Cl(G)) \mid G \in \mathscr{T}\}$ covers A.

Theorem 2.10. *If each proper regularly closed set of X is nearly compact with respect to X, then X is nearly compact.*

Proof. Let $\mathscr{S} = \{A_\alpha \mid \alpha \in \Delta\}$ be a pre-open cover of X. Since $\mathscr{S}$ is a cover of X, there exits an $A \in \mathscr{S}$ such that $A \neq \emptyset$. By Lemma 2.1, $Int(Cl(A))$ is regularly open in X, and so $X - Int(Cl(A))$ is regularly closed in X. By the assumption, we get a finite subcollection $\{A_{\alpha_k} \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\}$ such that

$$X - Int(Cl(A)) \subset \bigcup_{k=1}^{n} Int(Cl(A_{\alpha_k}))$$

and thus

$$X \subset \bigcup_{k=1}^{n} (Int(Cl(A_{\alpha_k}))) \cup Int(Cl(A)).$$

Therefore, by Theorem 2.2, X is nearly compact. □

Recall that a nonempty collection $\mathscr{F}$ of nonempty subsets of a set X is called a filter base (see [13, p. 49]) if whenever $F_1, F_2 \in \mathscr{F}$, one has $F_3 \subset F_1 \cap F_2$ for some $F_3 \in \mathscr{F}$. A filter base is called maximal (see [13, p. 47]) if it is not properly contained in another filter base. A filter base is always contained in a maximal filter base (see [13, p. 47]).

Definition 2.10. A filter base $\mathscr{F}$ on X is said to p-converge to a point $x \in X$ if, for each pre-open set A of X, with $x \in A$, there exists $F \in \mathscr{F}$ such that $F \subset Int(Cl(A))$.

Definition 2.11. A filter base $\mathscr{F}$ in X is said to p-accumulate to a point $x \in X$ if, for each pre-open set A of X with $x \in A$, one has $F \cap (Int(Cl(A))) \neq \emptyset$ for each $F \in \mathscr{F}$.

The following lemmas 2.4, 2.5, and 2.6 are easy to establish and hence their proofs are omitted.

Lemma 2.4. *If a filter base $\mathscr{F}$ in X p-converges to a point $x \in X$, then the filter base p-accumulates to x.*

Lemma 2.5. *Let $\mathscr{F}$ be a maximal filter base in X. Then $\mathscr{F}$ p-converges to $x \in X$ if and only if $\mathscr{F}$ p-accumulates to $x \in X$.*

Lemma 2.6. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two filter bases in X such that $\mathscr{F}_2$ is a subcollection of $\mathscr{F}_1$. Then $\mathscr{F}_2$ p-accumulates to a point $x \in X$ if $\mathscr{F}_1$ p-accumulates to $x \in X$.*

Theorem 2.11. *The following statements are equivalent.*

(a) *X is nearly compact.*
(b) *Each maximal filter base p-converges in X.*
(c) *Each filter base p-accumulates to some $x_0 \in X$.*
(d) *For each family $\mathscr{F}$ of pre-closed sets with $\bigcap_{F\in\mathscr{F}} F = \emptyset$, there exists a finite subcollection $\mathscr{E}$ of $\mathscr{F}$ such that $\bigcap_{E\in\mathscr{E}} Cl(Int(E)) = \emptyset$.*

Proof. (a)$\Rightarrow$(b). Let $\mathscr{F} = \{A_\alpha \mid \alpha \in \Delta\}$ be a maximal filter base in X. Suppose for contradiction that $\mathscr{F}$ does not p-converge to a point of X. By Lemma 2.5, $\mathscr{F}$ does also not p-accumulate to a point of X. This means that, for each $x \in X$, there exist a pre-open set G_x containing x and $A_{\alpha(x)}, \alpha(x) \in \Delta$ such that $(Int(Cl(G_x))) \cap A_{\alpha(x)} = \emptyset$. Then $\mathscr{G} = \{G_x \mid x \in X\}$ is a pre-open cover of X. By Theorem 2.2, we obtain a finite subcollection $\{G_{x_1}, G_{x_2}, \ldots, G_{x_n}\}$ of $\mathscr{G}$ such that $\bigcup_{k=1}^n Int(Cl(G_{x_k})) = X$. Since $\mathscr{F}$ is a filter base, there exists a $A_0 \in \mathscr{F}$ such that $A_0 \subset \bigcap_{k=1}^n A_{x_k}$. So $(Int(Cl(G_{x_k}))) \cap A_0 = \emptyset$ for each $k \in \{1, 2, \ldots, n\}$. We see that

$$\begin{aligned} A_0 = X \cap A_0 &= \Big(\bigcup_{k=1}^n Int(Cl(G_{x_k}))\Big) \cap A_0 \\ &= \bigcup_{k=1}^n ((Int(Cl(G_{x_k}))) \cap A_0) = \emptyset, \end{aligned}$$

a contradiction to the fact that $A_0 \neq \emptyset$.

(b)$\Rightarrow$(c). Let $\mathscr{F}$ be a filter base in X. Then there exists a maximal filter base $\mathscr{E}$ containing $\mathscr{F}$ as a subcollection. By (b), the filter base $\mathscr{E}$ p-converges to some $x_0 \in X$. By Lemma 2.4, $\mathscr{E}$ p-accumulates to x_0 and thus, by Lemma 2.6, $\mathscr{F}$ p-accumulates to x_0.

(c)$\Rightarrow$(d). Let $\mathscr{S} = \{A_\alpha \mid \alpha \in \Delta\}$ be a collection of pre-closed subsets of X such that $\bigcap_{\alpha\in\Delta} A_\alpha = \emptyset$. Suppose for contradiction that for each finite subcollection Δ_0 of Δ, we have $\bigcap_{\alpha\in\Delta_0} Cl(Int(A_\alpha)) \neq \emptyset$. Set $F_{\Delta_0} = \bigcap_{\alpha\in\Delta_0} Cl(Int(A_\alpha))$. Let Λ be the collection of all finite subcollection of Δ. We write $\mathscr{F} = \{F_\lambda \mid \lambda \in \Lambda\}$ (each F_λ bears the meaning as of F_{Δ_0}). We see that $\mathscr{F}$ is a filter base in X. By (c), $\mathscr{F}$ p-accumulates to some point

$x_0 \in X$. This means that $F_\lambda \cap (Cl(Int(A))) \neq \emptyset$ for each $\lambda \in \Lambda$ and each pre-open set A of X containing x_0, in particular,

$$Cl(Int(A_\alpha)) \cap A \neq \emptyset \tag{2.1}$$

for each $\alpha \in \Delta$ and each pre-open set A containing x_0. As $\bigcap_{\alpha\in\Delta} A_\alpha = \emptyset$, we have $x_0 \notin \bigcap_{\alpha\in\Delta} A_\alpha$ and thus $x_0 \notin A_{\alpha_0}$ for some $\alpha_0 \in \Delta$. So we get a pre-open set $X - A_{\alpha_0}$ such that that $x_0 \in X - A_{\alpha_0}$. According to the construction, $Cl(Int(A_{\alpha_0})) \in \mathscr{F}$. Now

$$\begin{aligned}&(Int(Cl(X - A_{\alpha_0})) \cap (Cl(Int(A_{\alpha_0})))\\ &\quad = (X - Cl(Int(A_{\alpha_0}))) \cap (Cl(Int(A_{\alpha_0}))) = \emptyset,\end{aligned}$$

a contradiction to (2.1).

(d)$\Rightarrow$(a). Suppose that $\mathscr{S} = \{A_\alpha \mid \alpha \in \Delta\}$ is a pre-open cover of X. Then $\{X - A_\alpha \mid \alpha \in \Delta\}$ is a collection of pre-closed sets such that $\bigcap_{\alpha\in\Delta}(X - A_\alpha) = \emptyset$. By (d), we obtain a finite subcollection Δ_0 of Δ such that $\bigcap_{\alpha\in\Delta_0} Cl(Int(X-A_\alpha)) = \emptyset$, which in turn implies that $\bigcup_{\alpha\in\Delta_0} Int(Cl(A_\alpha)) = X$. So, by Theorem 2.2, X is nearly compact. □

Remark 1. A subset A of X is α-set (see [14]) if $A \subset Int(Cl(Int(A)))$. So a subset A of X is α-set if and only if there exists an open set G such that $A \subset G \subset Cl(Int(A))$. We agree to call a collection $\mathscr{S}$ of α-sets of X an α-collection. A collection $\mathscr{S}$ of subsets of X is said to be an α-cover of X if $\mathscr{S}$ is an α-collection and covers X. We also note here in tune with Definition 2.6 that corresponding to an α-collection, there exists an open super-collection $\mathscr{U}$ of $\mathscr{S}$. As earlier, we see that $\mathscr{U}$ is an open cover of X if $\mathscr{S}$ is an α-cover of X and we call $\mathscr{U}$ an open super-cover of $\mathscr{S}$. In tune with Definition 2.7, a topological space X is said to be α-compact if each α-cover of X has a finite open super-cover. Proceeding as in the proof of Theorem 2.1, we find that α-compact spaces are also equivalent to nearly compact spaces. It means that nearly compact spaces can be defined in various ways. Thus new light to covering properties of topological spaces throws through the research of this paper.

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