An entire function sharing fixed points with its linear differential polynomial

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Abstract. We study the uniqueness of entire functions, when they share a linear polynomial, in particular, fixed points, with their linear differential polynomials.

1. Definitions and results

Let $f$ be a nonconstant meromorphic function defined in the open complex plane $\mathbb{C}$, and let $a = a(z)$ be a polynomial. Let us denote by $E(a; f)$ and $\overline{E}(a; f)$ the set of zeros of $f - a$, counted with multiplicities, and the set of all distinct zeros of $f - a$, respectively. If $A \subset \mathbb{C}$, then we denote by $n_A(r, a; f)$ the number of zeros of $f - a$, counted with multiplicities, that lie in $\{z : |z| \leq r\} \cap A$. The corresponding integrated counting function is defined by

$$N_A(r, a; f) = \int_0^r \frac{n_A(t, a; f) - n_A(0, a; f)}{t} dt + n_A(0, a; f) \log r.$$

We also denote by $\overline{N}_A(r, a; f)$ the reduced counting functions of those zeros of $f - a$ that lie in $\{z : |z| \leq r\} \cap A$.

Clearly, if $A = \mathbb{C}$, then $N_A(r, a; f) = N(r, a; f)$ and $\overline{N}_A(r, a; f) = \overline{N}(r, a; f)$.

The standard definitions and notation of the value distribution theory are available in [1].

The uniqueness of an entire function sharing a nonzero finite value with its first two derivatives was considered by Jank et al. [2] in 1986. The following is their result.

Theorem A (see [2]). Let $f$ be a nonconstant entire function and let $a$ be a nonzero finite value. If $\overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \overline{E}(a; f^{(2)})$, then $f \equiv f^{(1)}$. 

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Considering \( f = e^{\omega z} + \omega - 1 \) and \( a = \omega \), where \( \omega \) is a \((k - 1)\)th imaginary root of unity and \( k(\geq 3) \) is an integer, Zhong [10] pointed out that in Theorem A one can not replace the second derivative by any higher order derivative. Under this context, Zhong [10] proved the following theorem.

**Theorem B** (see [10]). Let \( f \) be a nonconstant entire function and let \( a \) be a nonzero finite number. If \( E(a; f) = E(a; f^{(1)}) \) and \( E(a; f) \subset \overline{E}(a; f^{(n)}) \cap E(a; f^{(n+1)}) \) for \( n \geq 1 \), then \( f \equiv f^{(n)} \).

Considering a shared linear polynomial, Lahiri and Ghosh [3] extended Theorem A in the following manner.

**Theorem C** (see [3]). Let \( f \) be a nonconstant entire function and let \( a(z) = \alpha z + \beta \), where \( \alpha(\neq 0), \beta \) are constants. If \( E(a; f) \subset E(a; f^{(1)}) \subset E(a; f^{(2)}) \), then either \( f = \lambda e^z \) or \( f = \alpha z + \beta + (\alpha z + \beta - 2\alpha) \exp\left(\frac{2z + \beta - 2\alpha}{\alpha}\right) \), where \( \lambda(\neq 0) \) is a constant.

In 1999, Li [7] considered linear differential polynomials and proved the following result.

**Theorem D** (see [7]). Let \( f \) be a nonconstant entire function and \( L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)} \), where \( a_1, a_2, \ldots, a_n(\neq 0) \) are constants and \( a(\neq 0) \) is a finite number. If \( \overline{E}(a; f) = \overline{E}(a; f^{(1)}) \subset \overline{E}(a; L) \cap \overline{E}(a; L^{(1)}) \), then \( f \equiv f^{(1)} \equiv L \).

In this paper, we consider the uniqueness of an entire function that shares a linear polynomial with linear differential polynomials generated by it. For two subsets \( A \) and \( B \) of \( \mathbb{C} \), we denote by \( A \Delta B \) the set \((A - B) \cup (B - A)\), which is called the symmetric difference of the sets \( A \) and \( B \).

We now state the main result of the paper.

**Theorem 1.1.** Let \( f \) be a nonconstant entire function and \( L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)} \), where \( a_2, a_3, \ldots, a_n(\neq 0) \) are constants and \( n(\geq 2) \) is a positive integer. Also, let \( a(z) = \alpha z + \beta \), where \( \alpha(\neq 0), \beta \) are constants. Suppose that \( A = \overline{E}(a; f) \Delta \overline{E}(a; f^{(1)}) \) and \( B = \overline{E}(a; f^{(1)}) \setminus \{\overline{E}(a; L) \cap \overline{E}(a; L^{(1)})\} \).

If the conditions
(i) \( N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{\log T(r, f)\} \),
(ii) \( N_B(r, a; f^{(1)}) = S(r, f) \),
(iii) each common zero of \( f - a \) and \( f^{(1)} - a \) has the same multiplicity, are satisfied, then \( f = L = \lambda e^z \), where \( \lambda(\neq 0) \) is a constant.

Putting \( A = B = \emptyset \), we obtain the following corollary which improves Theorem B for \( n \geq 2 \).

**Corollary 1.1.** Let \( f \) be a nonconstant entire function and \( L = a_1 f^{(1)} + a_2 f^{(2)} + \cdots + a_n f^{(n)} \), where \( a_2, a_3, \ldots, a_n(\neq 0) \) are constants and \( n(\geq 2) \) is an integer. Also let \( a(z) = \alpha z + \beta \), where \( \alpha(\neq 0), \beta \) are constants. Suppose
that $E(a; f) = E(a; f^{(1)})$ and $\overline{E}(a; f^{(1)}) \subset \{ \overline{E}(a; L) \cap \overline{E}(a; L^{(1)}) \}$. Then $f = L = \lambda e^z$, where $\lambda (\neq 0)$ is a constant.

The following examples show that the hypotheses (i) and (ii) of Theorem 1.1 are essential.

**Example 1.1.** Let $f(z) = e^z$, $L = f^{(2)} + f^{(3)}$ and $a(z) = z$. Then clearly $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{ \log T(r, f) \}$ and $N_B(r, a; f^{(1)}) = T(r, f) + O(1) \neq S(r, f)$. Also we note that the hypothesis (iii) of Theorem 1.1 holds, but $f \neq L$.

**Example 1.2.** Let $f(z) = e^z + z^2$, $L = f^{(3)} + f^{(4)}$ and $a(z) = 2z$. Then clearly $N_A(r, a; f) + N_A(r, a; f^{(1)}) = T(r, e^z) + O(1) \neq O\{ \log T(r, f) \}$ and $N_B(r, a; f^{(1)}) = S(r, f)$. Since $E(a; f^{(1)}) = \emptyset$, we note that the hypothesis (iii) of Theorem 1.1 holds, but $f \neq L$.

We denote by $N_2(r, a; f)$ the counting function, counted with multiplicities, of the multiple zeros of $f - a$.

A related result concerning the derivatives of an entire function can be found in [4].

**2. Lemmas**

In this section, we present some lemmas.

**Lemma 2.1** (see [9]). Let $g$ be a transcendental entire function and let $\phi(\neq 0)$ be a meromorphic function satisfying $T(r, \phi) = S(r, g)$. Then

$$T(r, g) \leq C_n \{ N(r, 0; g) + \overline{N}(r, 0; g^{(n)} - \phi) \} + S(r, g),$$

where $C_n$ is a constant depending only on $n (\geq 1)$.

**Lemma 2.2.** Let $f$ be a transcendental entire function and let $a = a(z)$ be a meromorphic function satisfying $a - a^{(n)} \neq 0$ and $T(r, a) = S(r, f)$. Then

$$T(r, f) \leq C_n \{ N(r, a; f) + \overline{N}(r, a; f^{(n)}) \} + S(r, f),$$

where $C_n$ is a constant depending only on $n (\geq 1)$.

**Proof.** Putting $g = f - a$ and $\phi = a - a^{(n)}$ in Lemma 2.1, we obtain the result. \qed

**Lemma 2.3** (see [5]). Let $f$ be transcendental entire function of finite order and let $a = a(z) = \alpha z + \beta$, where $\alpha (\neq 0), \beta$ are constants. Suppose that $A = \overline{E}(a; f) \Delta \overline{E}(a; f^{(1)})$. If $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O\{ \log T(r, f) \}$ and each common zero of $f - a$ and $f^{(1)} - a$ have the same multiplicity, then $m(r, a; f) = m(r, \frac{1}{T-a}) = S(r, f)$.

To prove the following lemma, we adapt some techniques from [5].
Lemma 2.4. Let $f$ be a transcendental entire function and $a(z) = \alpha z + \beta (\not\equiv 0)$. Suppose that
\[
L = a_2 f^{(2)} + a_3 f^{(3)} + \cdots + a_n f^{(n)} \quad \text{and} \quad h = \frac{(a - a^{(1)}) L - a (f^{(1)} - a^{(1)})}{f - a},
\]
where $a_2, a_3, \ldots, a_n (\not\equiv 0)$ are constants. Further, suppose that
\[
A = \overline{E}(a; f) \setminus \overline{E}(a; f^{(1)}) \quad \text{and} \quad B = \overline{E}(a; f^{(1)}) \setminus \{ \overline{E}(a; L) \cap \overline{E}(a; L^{(1)}) \}.
\]
If the conditions
(i) $N_A(r, a; f) + N_B(r, a; f^{(1)}) = S(r, f)$,
(ii) each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity,
(iii) $h$ is transcendental entire or meromorphic,
hold, then $m(r, a; f^{(1)}) = m\left(r, \frac{1}{f^{(1)} - a}\right) = S(r, f)$.

Proof. Since $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$, we have that if $z_0$ is a common zero of $f - a$ and $f^{(1)} - a$ with multiplicity $q \geq 2$, then $z_0$ is a zero of $a - a^{(1)}$ with multiplicity $q - 1$. So
\[
N_2(r, a; f) \leq 2N(r, 0; a - a^{(1)}) + N_A(r, a; f) = S(r, f).
\]

Hence, by the hypothesis, we see that
\[
N(r, h) \leq N_A(r, a; f) + N_B\left(r, a; f^{(1)}\right) + N_2(r, a; f) + S(r, f)
= S(r, f).
\]
Since $m(r, h) = S(r, f)$, we have $T(r, h) = S(r, f)$.

Now, by a simple calculation we get
\[
f = a + \frac{1}{h} \left\{ (a - a^{(1)}) (L - a) - a \left( f^{(1)} - a \right) \right\}.
\]
Differentiating, we obtain
\[
f^{(1)} = a^{(1)} + \left( \frac{1}{h} \right)^{(1)} \left\{ (a - a^{(1)}) (L - a) - a \left( f^{(1)} - a \right) \right\}
+ \left( \frac{1}{h} \right) \left\{ a^{(1)} (L - a) + (a - a^{(1)}) \left( L^{(1)} - a^{(1)} \right) - a^{(1)} \left( f^{(1)} - a \right) \right\}
- a \left( f^{(2)} - a^{(1)} \right) \right\}.
\]
This implies
\[
\frac{1}{f^{(1)} - a} = \frac{\xi}{\xi} - \frac{1}{\xi} \left( \frac{a - a^{(1)}}{h} \right)^{(1)} \frac{L - a_2 a^{(1)}}{f^{(1)} - a} - \frac{a - a^{(1)}}{h \xi} \frac{L^{(1)}}{f^{(1)} - a}
+ \frac{a}{h \xi} \frac{f^{(2)} - a^{(1)}}{f^{(1)} - a},
\]
(2.1)
where
\[ \xi = 1 + \left( \frac{a}{h} \right)^{(1)} \quad \text{and} \quad \zeta = a^{(1)} - a - \left( \frac{a(a - a^{(1)})}{h} \right)^{(1)} + \left( \frac{a - a^{(1)}}{h} \right)^{(1)} a_2 a^{(1)}. \]

We now verify that \( \xi \not\equiv 0 \) and \( \zeta \not\equiv 0 \). If \( \xi \equiv 0 \), then \( 1 + \left( \frac{a}{h} \right)^{(1)} \equiv 0 \). Integrating, we get \( h = \frac{a}{(c - z)} \), where \( c \) is a constant. This implies a contradiction as \( h \) is transcendental.

If \( \zeta \equiv 0 \), then \( a^{(1)} - a - \left( \frac{a(a - a^{(1)})}{h} \right)^{(1)} + \left( \frac{a - a^{(1)}}{h} \right)^{(1)} a_2 a^{(1)} \equiv 0 \), and so
\[ (\alpha - \beta)z - \frac{\alpha^2}{2} + \alpha_2 = \frac{a(a - \alpha)}{h} - \frac{a_2 \alpha(a - \alpha)}{h}, \]
where \( \alpha_2 \) is a constant. Therefore,
\[ h = \frac{(\alpha z + \beta - \alpha)(\alpha z + \beta - a_2 \alpha)}{-\frac{\alpha^2}{2} + (\alpha - \beta)z + \alpha_2}, \]
which is a contradiction as \( h \) is transcendental.

Since clearly \( T(r, \xi) + T(r, \zeta) = S(r, f) \), from (2.1) we get
\[ m(r, a; f^{(1)}) = m \left( r, \frac{1}{f^{(1)} - a} \right) = S(r, f). \]
This proves the lemma. \( \square \)

**Lemma 2.5** (see [6], p. 58). Each solution of the differential equation
\[ a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_0 f = 0, \]
where \( a_0(\neq 0), a_1, \ldots, a_n(\neq 0) \) are polynomials, is an entire function of finite order.

**Lemma 2.6** (see [1], p. 47). Let \( f \) be a nonconstant meromorphic function and let \( a_1, a_2, a_3 \) be three distinct meromorphic functions satisfying \( T(r, a_\nu) = S(r, f) \) for \( \nu = 1, 2, 3 \). Then
\[ T(r, f) \leq N(r, 0; f - a_1) + N(r, 0; f - a_2) + N(r, 0; f - a_3) + S(r, f). \]

**Lemma 2.7** (see [8], p. 92). Let \( f_1, f_2, \ldots, f_n \) be meromorphic functions which are nonconstant except possibly for \( f_n \), where \( n \geq 3 \). If \( f_n \neq 0 \), \( \sum_{j=1}^n f_j \equiv 1 \), and
\[ \sum_{j=1}^n N(r, 0; f_j) + (n - 1) \sum_{j=1}^n N(r, \infty; f_j) < \{ \mu + o(1) \} T(r, f_k) \]
for \( k = 1, 2, \ldots, n - 1 \) and for some \( \mu(0 < \mu < 1) \), then \( f_n \equiv 1 \).
3. Proof of Theorem 1.1

Proof. First, we see that $f$ can not be a polynomial. We suppose that $f$ is a polynomial. Then $T(r, f) = O(\log r)$ and $N_A(r, a; f) + N_A(r, a; f^{(1)}) = O(\log T(r, f)) = S(r, f)$ imply $A = \emptyset$. Also $N_B(r, a; f^{(1)}) = S(r, f)$ implies $B = \emptyset$. Therefore,

$$E(a; f) = E\left( a; f^{(1)} \right) \quad \text{and} \quad \mathcal{E} \left( a; f^{(1)} \right) \subset \mathcal{E}(a, L) \cap \mathcal{E} \left( a; L^{(1)} \right).$$

Let the degree of $f$ be greater than 1. Then $\deg(f - a) > \deg(f^{(1)} - a)$. Since each common zero of $f - a$ and $f^{(1)} - a$ has the same multiplicity, this contradicts the fact that $E(a; f) = E \left( a; f^{(1)} \right)$.

Next, let $f = A_1 z + B_1$, where $A_1(\neq 0), B_1$ are constants. Then $f^{(1)} = A_1$ and $L \equiv L^{(1)} \equiv 0$. Now, $(A_1 - \beta)/\alpha$ is the only zero of $f^{(1)} - a$, and $-\beta/\alpha$ is the only zero of $L - a$. Consequently, $\mathcal{E} \left( a; f^{(1)} \right) \subset \mathcal{E}(a, L)$ implies that $(A_1 - \beta)/\alpha = -\beta/\alpha$ and so $A_1 = 0$, a contradiction. Therefore $f$ is a transcendental entire function.

Now

$$N_{(2)} \left( r, a; f^{(1)} \right) \leq N_A \left( r, a; f^{(1)} \right) + N_B \left( r, a; f^{(1)} \right)$$

$$+ N_{(2)} \left( r, a; f^{(1)} | f = a \right) + S(r, f) \quad (3.1)$$

$$= N_{(2)} \left( r, a; f^{(1)} | f = a \right) + S(r, f),$$

where $N_{(2)} \left( r, a; f^{(1)} | f = a \right)$ denotes the counting function (counted with multiplicities) of those multiple zeros of $f^{(1)} - a$, which are also zeros of $f - a$.

We note that a common zero of $f - a$ and $f^{(1)} - a$ of multiplicity $q(\geq 2)$ is a zero of $a - a^{(1)} = (f^{(1)} - a^{(1)}) - (f^{(1)} - a)$ with multiplicity $q - 1(\geq 1)$. Therefore,

$$N_{(2)} \left( r, a; f^{(1)} | f = a \right) \leq 2 N \left( r, 0; a - a^{(1)} \right) = S(r, f).$$

So, from (3.1) we get

$$N_{(2)} \left( r, a; f^{(1)} \right) = S(r, f). \quad (3.2)$$

First, we suppose that $L^{(1)} \neq f^{(1)}$. Then, using (3.2), we get by the hypothesis that

$$N \left( r, a; f^{(1)} \right) \leq N_B \left( r, a; f^{(1)} \right) + N \left( r, \frac{a}{a - \alpha}; \frac{L^{(1)}}{f^{(1)} - \alpha} \right) + S(r, f)$$

$$\leq T \left( r, \frac{L^{(1)}}{f^{(1)} - \alpha} \right) + S(r, f) = N \left( r, \frac{L^{(1)}}{f^{(1)} - \alpha} \right) + S(r, f) \quad (3.3)$$

$$\leq N \left( r, a; f^{(1)} \right) + S(r, f).$$
Again,
\[ m(r, a; f) \leq m \left( r, \frac{f^{(1)} - \alpha}{f - a} \frac{1}{f^{(1)} - \alpha} \right) \leq m \left( r, \alpha; f^{(1)} \right) + S(r, f) \]
\[ = T \left( r, f^{(1)} \right) - N \left( r, \alpha; f^{(1)} \right) + S(r, f) \]
\[ = m \left( r, f^{(1)} \right) - N \left( r, \alpha; f^{(1)} \right) + S(r, f) \]
\[ \leq m(r, f) - N \left( r, \alpha; f^{(1)} \right) + S(r, f) \]
\[ = T(r, f) - N \left( r, \alpha; f^{(1)} \right) + S(r, f), \]
and so
\[ N \left( r, \alpha; f^{(1)} \right) \leq N(r, a; f) + S(r, f). \]
Thus from (3.3) we get
\[ N \left( r, a; f^{(1)} \right) \leq N(r, a; f) + S(r, f). \]  
(3.4)

Again,
\[ N(r, a; f) \leq N_A(r, a; f) + N \left( r, a; f^{(1)} \mid f = a \right) \]
\[ \leq N \left( r, a; f^{(1)} \right) + S(r, f). \]  
(3.5)

Therefore, from (3.4) and (3.5), we deduce that
\[ N \left( r, a; f^{(1)} \right) = N(r, a; f) + S(r, f). \]  
(3.6)

Let \( h \), defined as in Lemma 2.4, be transcendental. Then
\[ T(r, f) = m(r, f) \leq m \left( r, \frac{1}{h} \left\{ \left( a - a^{(1)} \right) L - af^{(1)} \right\} \right) + S(r, f) \]
\[ \leq m \left( r, f^{(1)} \right) + m \left( r, \left( a - a^{(1)} \right) \frac{L}{f^{(1)}} - a \right) + S(r, f) \]
\[ \leq m \left( r, f^{(1)} \right) + S(r, f) = T \left( r, f^{(1)} \right) + S(r, f) \]
\[ = m \left( r, f^{(1)} \right) + S(r, f) \leq m(r, f) + S(r, f) = T(r, f) + S(r, f). \]
Therefore,
\[ T \left( r, f^{(1)} \right) = T(r, f) + S(r, f). \]  
(3.7)

Again, by Lemma 2.4 we get \( m \left( r, a; f^{(1)} \right) = S(r, f) \). Then, from (3.6) and (3.7), we have that
\[ m(r, a; f) + m \left( r, a; f^{(1)} \right) = S(r, f). \]  
(3.8)
Next we suppose that \( h \) is rational. Then by Lemma 2.5 we see that \( f \) is of finite order. So, by the hypothesis and Lemma 2.3, we get the equality
\[ m(r, a; f) = S(r, f). \]

Since
\[ T(r, f^{(1)}) = m(r, f^{(1)}) \leq m(r, f) + S(r, f) = T(r, f) + S(r, f), \]
from (3.6) we get
\[ m(r, a; f^{(1)}) \leq m(r, a; f) + N(r, a; f) - N(r, a; f^{(1)}) + S(r, f) = S(r, f). \]

Hence in this case also we obtain (3.8).

We now put
\[ \phi = \frac{f^{(1)} - a}{f - a} \quad \text{and} \quad \psi = \frac{L - a}{f^{(1)} - a}. \]

Then by (3.8) we get \( m(r, \phi) + m(r, \psi) = S(r, f) \). Also, from the hypothesis we have
\[ N(r, \phi) \leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_{(2)}(r, a; f) + S(r, f) = S(r, f), \]

because
\[ N_{(2)}(r, a; f) \leq N_A(r, a; f) + 2N(r, 0; a - a^{(1)}) + S(r, f) = S(r, f). \]

Again, by (3.2) and the hypothesis, we get
\[ N(r, \psi) \leq N_A(r, a; f) + N_B(r, a; f^{(1)}) + N_{(2)}(r, a; f^{(1)}) + S(r, f) = S(r, f). \]

Therefore,
\[ T(r, \phi) + T(r, \psi) = S(r, f). \quad (3.9) \]

Let \( z_1 \) be a simple zero of \( f - a \) such that \( z_1 \notin A \cup B \) and \( a(z_1) - a^{(1)}(z_1) \neq 0 \). Then \( f(z_1) = f^{(1)}(z_1) = L(z_1) = L^{(1)}(z_1) = a(z_1) \). Now, by Taylor’s expansion in some neighbourhood of \( z_1 \), we get
\[
\begin{align*}
f(z) - a(z) &= (f - a)(z_1) + (f - a)^{(1)}(z_1)(z - z_1) + O(z - z_1)^2 \\
&= \left(a(z_1) - a^{(1)}(z_1)\right)(z - z_1) + O(z - z_1)^2,
\end{align*}
\]
\[
\begin{align*}
f^{(1)}(z) - a(z) &= \left(f^{(1)} - a\right)(z_1) + \left(f^{(1)} - a\right)^{(1)}(z_1)(z - z_1) + O(z - z_1)^2 \\
&= \left(f^{(2)}(z_1) - a^{(1)}(z_1)\right)(z - z_1) + O(z - z_1)^2
\end{align*}
\]
and
\[
\begin{align*}
L(z) - a(z) &= (L - a)(z_1) + (L - a)^{(1)}(z_1)(z - z_1) + O(z - z_1)^2 \\
&= \left(a(z_1) - a^{(1)}(z_1)\right)(z - z_1) + O(z - z_1)^2.
\end{align*}
\]
Therefore, in a neighbourhood of $z_1$, we obtain

$$
\phi(z) = \frac{\left\{ f^{(2)}(z_1) - a^{(1)}(z_1) \right\} (z - z_1) + O(z - z_1)^2}{(a(z_1) - a^{(1)}(z_1)) (z - z_1) + O(z - z_1)^2}
$$

$$
= \frac{f^{(2)}(z_1) - \alpha + O(z - z_1)}{a(z_1) - \alpha + O(z - z_1)} = \frac{f^{(2)}(z_1) - \alpha}{a(z_1) - \alpha} + O(z - z_1)
$$

(3.10)

and

$$
\psi(z) = \frac{(a(z_1) - a^{(1)}(z_1)) (z - z_1) + O(z - z_1)^2}{(f^{(2)}(z_1) - a^{(1)}(z_1)) (z - z_1) + O(z - z_1)^2}
$$

$$
= \frac{a(z_1) - \alpha + O(z - z_1)}{f^{(2)}(z_1) - \alpha + O(z - z_1)} = \frac{a(z_1) - \alpha}{f^{(2)}(z_1) - \alpha} + O(z - z_1).
$$

(3.11)

We put $M = \psi - 1/\phi$. Then from (3.9) we get $T(r, M) = S(r, f)$. Also, in some neighbourhood of $z_1$, we have, by (3.10) and (3.11), that $M(z) = O(z - z_1)$.

If $M \equiv 0$, then

$$
\overline{N}(r; a; f) \leq N_A(r, a; f) + N_B \left( r, a; f^{(1)} \right) + N_2(r, a; f)
$$

$$
+ N(r, 0; a - a^{(1)}) + N(r, 0; M)
$$

$$
= S(r, f),
$$

and so, by (3.6) and Lemma 2.2, we have $T(r, f) = S(r, f)$, a contradiction. Thus $M \equiv 0$ and so

$$
L \equiv f.
$$

(3.12)

Differentiating (3.12) we get $L^{(1)} \equiv f^{(1)}$, which contradicts our hypothesis that $L^{(1)} \neq f^{(1)}$. Therefore, indeed we have $L^{(1)} \equiv f^{(1)}$.

Next we suppose that $L^{(1)} \neq L$. Then, by the hypothesis and (3.2), we get

$$
N \left( r, a; f^{(1)} \right) \leq N_B \left( r, a; f^{(1)} \right) + N \left( r, 1; \frac{L^{(1)}}{L} \right) + S(r, f)
$$

$$
\leq T \left( r, \frac{L^{(1)}}{L} \right) + S(r, f) = N \left( r, \frac{L^{(1)}}{L} \right) + S(r, f)
$$

(3.13)

$$
= \overline{N}(r, 0; L) + S(r, f).
$$

Again,

$$
m(r, a; f) = m \left( r, \frac{L}{f - a} \right) \leq m(r, 0; L) + S(r, f)
$$

$$
= T(r, L) - N(r, 0; L) + S(r, f) = m(r, L) - N(r, 0; L) + S(r, f)
$$
\[
\begin{align*}
\leq m\left(\frac{r, L}{f}\right) + m(r, f) - N(r, 0; L) + S(r, f) \\
= m(r, f) - N(r, 0; L) + S(r, f) = T(r, f) - N(r, 0; L) + S(r, f)
\end{align*}
\]

and so
\[
N(r, 0; L) \leq N(r, a; f) + S(r, f).
\]

Now, by (3.13) we get
\[
N\left(r, a; f^{(1)}\right) \leq N\left(r, a; f\right) + S(r, f) \tag{3.14}
\]

Also,
\[
N\left(r, a; f\right) \leq N_A\left(r, a; f\right) + N\left(r, a; f^{(1)} \mid f = a\right) \leq N\left(r, a; f^{(1)}\right) + S(r, f) \tag{3.15}
\]

From (3.14) and (3.15) we get (3.6).

Now, using Lemmas 2.3–2.5 and (3.6), we similarly obtain (3.8). Further, using \(\phi\) and \(\psi\) and proceeding likewise, we get (3.12).

Solving \(L - f \equiv 0\), we find that
\[
f = c_1 e^{\alpha_1 z} + c_2 e^{\alpha_2 z} + \cdots + c_k e^{\alpha_k z}, \tag{3.16}
\]

where \(\alpha_1, \alpha_2, \ldots, \alpha_k\) are the roots of \(\sum_{j=2}^{n} a_j z^j = 1\) and \(c_1, c_2, \ldots, c_k\) are constants or polynomials, not all identically zero, and \(k(\leq n)\) is an integer.

Differentiating (3.16), we get
\[
f^{(1)} = \left(c_1^{(1)} + c_1 \alpha_1\right) e^{\alpha_1 z} + \left(c_2^{(1)} + c_2 \alpha_2\right) e^{\alpha_2 z} + \cdots + \left(c_k^{(1)} + c_k \alpha_k\right) e^{\alpha_k z}. \tag{3.17}
\]

From (3.16), (3.17), and \(\phi = \left(f^{(1)} - a\right)/(f - a)\), we get
\[
\begin{align*}
\left(\phi c_1 - c_1^{(1)} - c_1 \alpha_1\right) e^{\alpha_1 z} + \left(\phi c_2 - c_2^{(1)} - c_2 \alpha_2\right) e^{\alpha_2 z} + \cdots + \\
+ \left(\phi c_k - c_k^{(1)} - c_k \alpha_k\right) e^{\alpha_k z} &\equiv a(\phi - 1).
\end{align*}
\]

We suppose that \(\phi \neq 1\). Then, from the above, we have
\[
\sum_{j=1}^{k} \frac{\phi c_j - c_j^{(1)} - c_j \alpha_j}{a(\phi - 1)} e^{\alpha_j z} = 1. \tag{3.18}
\]

We note that \(T(r, f) = O(T(r, e^{\alpha_j z}))\) for \(j = 1, 2, \ldots, k\).

If the left hand side of (3.18) contains more than two terms, then by Lemma 2.7 we get
\[
\frac{\phi c_j - c_j^{(1)} - c_j \alpha_j}{a(\phi - 1)} e^{\alpha_j z} \equiv 1 \tag{3.19}
\]

for one value of \(j \in \{1, 2, \ldots, k\}\). From (3.19) we see that \(T(r, e^{\alpha_j z}) = S(r, f) = S(r, e^{\alpha_j z})\), a contradiction.
We now suppose that the left hand side of (3.18) contains only two terms, say,
\[
\phi c_j - c_j^{(1)} - c_j \alpha_j \frac{a}{a(\phi - 1)} e^{\alpha_j z} + \phi c_l - c_l^{(1)} - c_l \alpha_l \frac{a}{a(\phi - 1)} e^{\alpha_l z} \equiv 1.
\]
By Lemma 2.6 we get
\[
T(r, e^{\alpha_j z}) \leq N(r, 0; e^{\alpha_j z}) + N(r, \infty; e^{\alpha_j z})
\]
\[
+ N\left(r, \frac{a(\phi - 1)}{\phi c_j - c_j^{(1)} - c_j \alpha_j}; e^{\alpha_j z}\right) + S(r, e^{\alpha_j z})
\]
\[
= N(r, 0; e^{\alpha_j z}) + S(r, e^{\alpha_j z}) = S(r, e^{\alpha_j z}),
\]
a contradiction.

Finally, we suppose that the left hand side of (3.18) contains only one term, say,
\[
\phi c_j - c_j^{(1)} - c_j \alpha_j \frac{a}{a(\phi - 1)} e^{\alpha_j z} \equiv 1.
\]
Then \(T(r, e^{\alpha_j z}) = S(r, f) = S(r, e^{\alpha_j z})\), a contradiction.

Therefore, \(\phi \equiv 1\) and so \(f^{(1)} \equiv f\). Hence, by (3.12) we get \(L \equiv L^{(1)}\), a contradiction to the supposition. Thus, indeed, we have \(L \equiv L^{(1)}\).

Now \(L \equiv L^{(1)} \equiv f^{(1)}\) implies \(L = L^{(1)} = f^{(1)} = \lambda e^z\), where \(\lambda \neq 0\) is a constant. Therefore \(f = \lambda e^z + K\), where \(K\) is a constant. By Lemma 2.6 we get
\[
T(r, \lambda e^z) \leq N(r, 0; \lambda e^z) + N(r, \infty; \lambda e^z) + N(r, a - K; \lambda e^z) + S(r, \lambda e^z)
\]
\[
= N(r, a; f) + S(r, \lambda e^z),
\]
which implies \(N(r, a; f) \neq S(r, f)\). Again, since
\[
N_A(r, a; f) + N_B\left(r, a; f^{(1)}\right) = S(r, f),
\]
we get
\[
\overline{E}(a; f) \cap \overline{E}\left(a; f^{(1)}\right) \neq \emptyset.
\]
But this implies \(K = 0\) and so \(f = L = \lambda e^z\). The proof is complete. \(\square\)

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References


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