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Equivalence results for implicit Jungck–Kirk type iterations

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Abstract. We show that the implicit Jungck–Kirk-multistep, implicit Jungck–Kirk–Noor, implicit Jungck–Kirk–Ishikawa, and implicit Jungck–Kirk–Mann iteration schemes are equivalently used to approximate the common fixed points of a pair of weakly compatible generalized contractive-like operators defined on normed linear spaces. Our results contribute to the existing results on the equivalence of fixed point iteration schemes by extending them to pairs of maps. An example to show the applicability of the main results is included.

1. Introduction

The concept of employing various iterative schemes in approximating fixed points of contractive-like operators are very useful in fixed point theory and applications, and other relevant fields like numerical analysis, operation research etc. This is due to the close relationship that exists between the problem of solving nonlinear equations and that of approximating fixed points of the corresponding contractive-like operators. An example can be found in Glowinski and Le-Tallec [7], where a three-step iteration process is used to solve elastoviscoplasticity, liquid crystal, and eigenvalue problems. Haubruge et al. [8] studied convergence analysis of three-step iterative processes in [7], and applied the scheme to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming, and minimization of sums of convex functions. The authors [8] also proved that three-step iteration leads to highly parallelized algorithms under certain conditions. Thus, we can say that multistep iterative schemes play important role in solving various problems in pure and applied sciences.

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In the year 1965, Kirk [9] initiated an iterative scheme in the form of series and proved a fixed point theorem for mappings which do not increase distances. The scheme is defined as follows.

Definition 1.1 (see [9])**.** Let $(E, \|.\|)$ be a normed linear space, let D be a non-empty convex closed subset of E, let $T : D \to D$ be a selfmap of D, and let $x_0 \in E$. The sequence of iterations $\{x_n\}_{n=1}^{\infty}$ is defined by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i T^i x_n, \quad n \geq 0, \ \sum_{i=0}^{k} \alpha_i = 1. \tag{1}$$

The following iteration scheme was introduced by Mann [10] to establish mean value methods in iteration.

Definition 1.2 (see [10])**.** Let E and D be the same as in Definition 1.1. For any given $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ of Mann iterations is defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n = 1, 2, \ldots, \tag{2}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in $[0, 1)$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Several authors have written papers on Kirk type iterative schemes, worthy of mention are the following: the explicit Kirk–Mann [15], explicit Kirk–Ishikawa [15], explicit Kirk–Noor [4], and explicit Kirk-multistep [1] iterative schemes. For example, Olatinwo [13] gave the following Kirk–Mann iterative scheme and prove a stability result.

Definition 1.3 (see [13])**.** Let E and D be the same as in Definition 1.1, and let $x_0 \in E$. The sequence $\{x_n\}_{n=0}^{\infty}$ of iterations is defined by

$$x_{n+1} = \alpha_{n,0} x_n + \sum_{i=1}^{q_1} \alpha_{n,i} T^i x_n, \ \sum_{i=1}^{q_1} \alpha_{n,i} = 1, \quad n = 1, 2, \ldots, \tag{3}$$

where $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\alpha_{n,i} \in [0, 1)$, such that $\sum_{n=0}^{\infty} \alpha_{n,i} = \infty$, and q_1 is a fixed integer.

In 2014, Akewe et al. [1] introduced an explicit Kirk-multistep iterative scheme, proved strong convergence and stability results for contractive-like operators in normed linear spaces, and gave useful numerical examples to back up their schemes.

Implicit iterations have advantage over explicit iterations for nonlinear problems as they provide better approximation of fixed points, and are widely used in many applications, when explicit iterations are inefficient. Approximation of fixed points in computer oriented programs using implicit iterations can reduce the computational cost of the fixed point problems (see [5]). Many researchers have proved useful results on the equivalence of the various iterative schemes, that is, the convergence of any of the iterative method to the unique fixed point of the contractive operator for a single map T is equivalent to the convergence of the other iterative schemes (see [4], [5], [7], [12], and [15]). However, it is observed that little is known about the equivalence of implicit schemes for pair of maps. This work will address these areas.

In [3], the authors introduced an implicit Jungck–Kirk type iterative scheme and used it to approximate the unique common fixed point of a pair of weakly compatible generalized contractive-like operators defined on a Banach space, and gave an example to demonstrate the application of the convergence results.

Let E be a Banach space and let $S, T : E \to E$ be non-selfcommuting maps of E with $T(E) \subseteq S(E)$. We shall present some of these implicit Jungck–Kirk type iterative schemes to establish equivalence results.

Definition 1.4 (see [2])**.** Let $u_0 \in E$. The implicit Jungck–Kirk–Mann iteration is the sequence $\{Su_n\}_{n=0}^{\infty}$ defined by

$$Su_{n+1} = \alpha_{n,0} Sx_n^1 + \sum_{i=1}^{q_1} \alpha_{n,i} T^i x_{n+1}, \quad \sum_{i=0}^{q_1} \alpha_{n,i} = 1, \quad n = 0, 1, \dots, \tag{4}$$

where $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\alpha_{n,i} \in [0, 1]$, and q_1 is a fixed integer.

Definition 1.5 (see [2])**.** Let $z_0 \in E$. The implicit Jungck–Kirk–Ishikawa iteration is the sequence $\{Sz_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} Sz_{n+1} &= \alpha_{n,0} Sz_n^1 + \sum_{i=1}^{q_1} \alpha_{n,i} T^i z_{n+1}, \quad \sum_{i=0}^{q_1} \alpha_{n,i} = 1, \\ Sz_n^1 &= \beta_{n,0}^1 Sz_n + \sum_{i=1}^{q_2} \beta_{n,i}^1 T^i z_n^1, \quad \sum_{i=0}^{q_2} \beta_{n,i}^1 = 1, \quad n = 0, 1, dots, \end{aligned} \tag{5}$$

where $q_1 \geq q_2$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta_{n,i}^1 \geq 0$, $\beta_{n,0}^1 \neq 0$, $\alpha_{n,i}, \beta_{n,i}^1 \in [0, 1)$, such that $\sum_{n=0}^{\infty} \alpha_{n,i} = \infty$, and q_1, q_2 are fixed integers.

Definition 1.6 (see [2])**.** Let $y_0 \in E$. The implicit Jungck–Kirk–Noor iteration is the sequence $\{Sy_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} Sy_{n+1} &= \alpha_{n,0} Sy_n^1 + \sum_{i=1}^{q_1} \alpha_{n,i} T^i y_{n+1}, \quad \sum_{i=0}^{q_1} \alpha_{n,i} = 1, \\ Sy_n^1 &= \beta_{n,0}^1 Sy_n^2 + \sum_{i=1}^{q_2} \beta_{n,i}^1 T^i y_n^1, \quad \sum_{i=0}^{q_2} \beta_{n,i}^1 = 1, \\ Sy_n^2 &= \beta_{n,0}^2 Sy_n + \sum_{i=1}^{q_3} \beta_{n,i}^2 T^i y_n^2, \quad \sum_{i=0}^{q_3} \beta_{n,i}^2 = 1, \quad n = 0, 1, \dots, \end{aligned} \tag{6}$$

where $q_1 \geq q_2 \geq q_3$, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta^1_{n,i} \geq 0$, $\beta^1_{n,0} \neq 0$, $\beta^2_{n,i} \geq 0$, $\beta^2_{n,0} \neq 0$, $\alpha_{n,i}, \beta^1_{n,i}, \beta^2_{n,i} \in [0,1)$, such that $\sum_{n=0}^{\infty} \alpha_{n,i} = \infty$, and q_1, q_2 and q_3 are fixed integers.

Definition 1.7 (see [2])**.** Let $x_0 \in E$. The implicit Jungck–Kirk-multistep iteration is the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$\begin{aligned} Sx_{n+1} &= \alpha_{n,0} Sx_n^1 + \sum_{i=1}^{q_1} \alpha_{n,i} T^i x_{n+1}, \quad \sum_{i=0}^{q_1} \alpha_{n,i} = 1, \\ Sx_n^l &= \beta^l_{n,0} Sx_n^{l+1} + \sum_{i=1}^{q_{l+1}} \beta^l_{n,i} T^i x_n^l, \quad \sum_{i=0}^{q_{l+1}} \beta^l_{n,i} = 1, \quad l = 1, 2, \ldots, k-2, \\ Sx_n^{k-1} &= \beta^{k-1}_{n,0} Sx_n + \sum_{i=1}^{q_k} \beta^{k-1}_{n,i} T^i x_n^{k-1}, \quad \sum_{i=0}^{q_k} \beta^{k-1}_{n,i} = 1, \quad k \geq 2, \end{aligned} \tag{7}$$

where $n = 0, 1, \ldots,$ $q_1 \geq q_2 \geq q_3 \geq \cdots \geq q_k$ for each j, $\alpha_{n,i} \geq 0$, $\alpha_{n,0} \neq 0$, $\beta^l_{n,i} \geq 0$, $\beta^l_{n,0} \neq 0$ for each l, $\alpha_{n,i}, \beta^l_{n,i} \in [0,1)$ for each l such that $\sum_{n=0}^{\infty} \alpha_{n,i} = \infty$, and q_1, q_l are fixed integers (for each l).

Remark 1.8. (i) The elements u_n, z_n, y_n and x_n in (4), (5), (6), and (7), respectively, are usually evaluated using Mann iterations (2).

(ii) The implicit Jungck–Kirk-multistep iteration (7) is an important generalization of the implicit Jungck–Kirk–Noor (6), implicit Jungck–Kirk–Ishikawa (5), and implicit Jungck–Kirk–Mann (4) iterations, because one can recover (6), (5), and (4) from (7). In fact, if k=3 in (7), then we get the implicit Jungck–Kirk–Noor iteration (6). If k=2 in (7), then we get the implicit Jungck–Kirk–Ishikawa iteration (5), and if k=2 and $q_2 = 0$ in (7), then we get the implicit Jungck–Kirk–Mann iteration (4).

The concepts of coincidence points, commuting maps, and weakly compatible maps are useful in showing the link beween two and multivalue maps in relation to fixed point iteration procedures. Thus we need the following definition, example, and lemmas in proving our results.

Definition 1.9. Let E be a Banach space and let Y be an arbitrary set. Let $S, T : Y \to E$ be two non-self mappings such that $T(Y) \subseteq S(Y)$. A point $p \in Y$ is called a coincident point of a pair of self maps S, T if there exists a point q (called a point of coincidence) in E such that $q = Sp = Tp$. Self maps S and T are said to be weakly compatible if they commute at their coincidence points, that is, if $Sp = Tp$ for some $p \in Y$, then $STp = TSp$.

The next example, given by Djoudi and Alouche [6], shows that weakly compatible maps are more general than those with other compatibility type maps. For instance, if two maps S and T are compatible, compatible of type (A), compatible of type (B), compatible of type (P), compatible of type (C), then they are weakly compatible, but the converse is not true in general.

Example 1.10 (see [6], Example 1.3)**.** Let $(X,d)=([0,10],|\cdot|)$. Define S and T by

$$Sx=\begin{cases}3, & \text{if } x\in(0,2],\\ 0, & \text{if } x\in\{0\}\cup(2,10]\end{cases} \quad \text{and} \quad Tx=\begin{cases}0, & \text{if } x=0,\\ x+8, & \text{if } x\in(0,2],\\ x-2, & \text{if } x\in(2,10].\end{cases}$$

Then S, T are weakly compatible, but are not compatible of type (A), type (B), type (P), type (C), and are also not compatible. For details of the proof see [3].

Different generalized contractive-like operators can be found in literature (see [5]), some of them that are relevant to this study are defined as follows.

Definition 1.11 (see [11])**.** Let E be a Banach space and let Y be an arbitrary set. The maps $S,T:Y\to E$ with $T(Y)\subseteq S(Y)$ are called the generalized Zamfirescu operators if

$$\|Tx-Ty\|\le\delta\|Sx-Sy\|+2\delta\|Sx-Tx\|,\quad x,y\in Y, \tag{8}$$

where $\delta\in[0,1)$.

Olatinwo [12] considered the contractive condition

$$\|Tx-Ty\|\le\delta\|Sx-Sy\|+\varphi(\|Sx-Tx\|),\quad x,y\in Y, \tag{9}$$

where $\delta\in[0,1)$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a monotone increasing function such that $\varphi(0)=0$.

Clearly, if $\varphi(x)=2\delta x$ in (9), then we get (8). Thus the contractive condition (9) generalizes (8).

We shall now employ a similar contractive condition in one of the following lemmas for proving our results.

Lemma 1.12 (see [14])**.** *Let $\{a_n\}_{n=0}^\infty$ be a nonnegative real sequence satisfying the inequality $a_{n+1}\le(1-\lambda_n)a_n+e_n$, where $\lambda_n\in(0,1)$ for all $n\ge n_0$, $\sum_{n=0}^\infty\lambda_n=\infty$, and $e_n=o(\lambda_n)$. Then $\lim_{n\to\infty}a_n=0$.*

Lemma 1.13. *Let $(E,\|\cdot\|)$ be a normed linear space and let $S,T:Y\to E$ be nonself commuting maps satisfying* (9) *such that $T(Y)\subseteq S(Y)$ and*

$$\|S^2x-T(Sx)\|\le\|Sx-Tx\|,\quad \|S^2x-Sy\|\le\|Sx-Sy\| \quad \textit{for all } x,y\in Y.$$

Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a sublinear monotone increasing function such that $\varphi(0)=0$. Let w be the coincident point of S, T, S^i, T^i (i.e., $Sw=Tw=p$ and $S^iw=T^iw=p$). Then, for all $x,y\in Y$,

$$\|T^ix-T^iy\|\le\delta^i\|Sx-Sy\|+\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j(\|Sx-Tx\|). \tag{10}$$

Proof. It is not difficult to see that if φ is subadditive, then each power φ^j is also subadditive.

We prove the inequality (10) by induction on i as follows. For $i = 1$, the inequality (10) reduces to our assumption (9). Now, we suppose that (10) is true for $i = n$, i.e.,

$$\|T^n x - T^n y\| \leq \delta^n \|Sx - Sy\| + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j} \varphi^j (\|Sx - Tx\|), \tag{11}$$

and then we show that (10) holds for $i = n+1$. Using (11) and the assumptions of the lemma, we have

$$\begin{aligned}
\|T^{n+1}x - T^{n+1}y\| &\leq \delta^n \|S(Tx) - S(Ty)\| + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j} \varphi^j (\|S(Tx) - T^2 x\|) \\
&\leq \delta^n \left(\varphi\left(\|S^2 x - T(Sx)\|\right) + \delta \|S^2 x - S^2 y\|\right) \\
&\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j} \varphi^j \left(\varphi\left(\|S^2 x - T(Sx)\|\right) + \delta \|S^2 x - S(Tx)\|\right) \\
&\leq \delta^n \left(\varphi\left(\|Sx - Tx\|\right) + \delta \|Sx - Sy\|\right) \\
&\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j} \varphi^j \left(\varphi(\|Sx - Tx\|) + \delta \|Sx - Tx\|\right) \\
&\leq \delta^{n+1} \|Sx - Sy\| + \delta^n \varphi(\|Sx - Tx\|) \\
&\quad + \sum_{j=1}^{n} \binom{n}{j} \delta^{n+1-j} \varphi^j (\|Sx - Tx\|) + \sum_{j=1}^{n} \binom{n}{j} \delta^{n-j} \varphi^{j+1} (\|Sx - Tx\|) \\
&= \delta^{n+1} \|Sx - Sy\| + \sum_{j=1}^{n+1} \binom{n+1}{j} \delta^{n+1-j} \varphi^j (\|Sx - Tx\|).
\end{aligned}$$

Therefore, (10) holds for $i = n+1$. The proof is complete. □

Next, we prove that the convergences of the various implicit Jungck–Kirk type iterative schemes (4), (5), (6), and (7) are equivalent for a pair of weakly compatible maps S, T.

2. Main results

Theorem 2.1. *Let $(E, \|\cdot\|)$ be a normed linear space and let $S, T : E \to E$, $T(E) \subseteq S(E)$, be two weakly compatible mappings satisfying the generalized contractive-like condition* (10)*, where φ is a sublinear monotone increasing function such that $\varphi(0) = 0$. Let p be the unique common fixed point of S, T, S^i, T^i (i.e., $Sp = Tp = p$ and $S^i p = T^i p = p$). If $u_0 = x_0 \in E$, then the following statements are equivalent:*

(i) *Implicit Jungck–Kirk–Mann iteration* (4) *converges strongly to* p;
(ii) *Implicit Jungck–Kirk-multistep iteration* (7) *converges strongly to* p.

Proof. (i) $\Rightarrow$ (ii). Assume that $\lim_{n\to\infty} Su_n = p$. Then, using (4), (7), and the contractive condition (10)), we get

$$\begin{aligned}\|Su_{n+1} - Sx_{n+1}\| &\leq \alpha_{n,0}\|Su_n - Sx_n^1\| + \sum_{i=1}^{q_1} \alpha_{n,i}\|T^i u_{n+1} - T^i x_{n+1}\| \\ &\leq \alpha_{n,0}\|Su_n - Sx_n^1\| + \sum_{i=1}^{q_1} \alpha_{n,i}\delta^i \|Su_{n+1} - Sx_{n+1}\| \\ &\quad + \sum_{i=1}^{q_1} \alpha_{n,i} \sum_{j=0}^{i} \binom{i}{j} \delta^{i-j}\varphi^j (\|Su_{n+1} - T^i u_{n+1}\|).\end{aligned} \tag{12}$$

From (12), we have

$$\begin{aligned}\|Su_{n+1} - T^i u_{n+1}\| &= \|Su_{n+1} - Sp + Tp - T^i u_{n+1}\| \\ &\leq \|Su_{n+1} - p\| + \|Tp - T^i u_{n+1}\| \leq \|Su_{n+1} - p\| + \delta^i \|p - Su_{n+1}\| \\ &\quad + \sum_{i=1}^{q_1} \alpha_{n,i} \sum_{j=0}^{i} \binom{i}{j} \delta^{i-j}\varphi^j \|Sp - Tp\| \\ &= \left(1 + \delta^i\right) \|Su_{n+1} - p\|.\end{aligned} \tag{13}$$

Substituting (13) in (12), we get

$$\begin{aligned}\|Su_{n+1} - Sx_{n+1}\| &\leq \frac{\alpha_{n,0}}{1 - \sum_{i=1}^{q_1} \alpha_{n,i}\delta^i} \left\|Su_n - Sx_n^1\right\| \\ &\quad + \frac{\sum_{i=1}^{q_1} \alpha_{n,i} \sum_{j=0}^{i} \binom{i}{j} \delta^{i-j}\varphi^j \left(1 + \delta^i\right) \|Su_{n+1} - p\|}{1 - \sum_{i=1}^{q_1} \alpha_{n,i}\delta^i}.\end{aligned} \tag{14}$$

Using (4), (7), and (10), we have

$$\begin{aligned}\left\|Su_n - Sx_n^1\right\| &\leq \beta_{n,0}^1 \left\|Su_n - Sx_n^2\right\| + \sum_{i=1}^{q_2} \beta_{n,i}^1 \left\|Su_n - T^i u_n + T^i u_n - T^i x_n^1\right\| \\ &\leq \beta_{n,0}^1 \left\|Su_n - Sx_n^2\right\| + \sum_{i=1}^{q_2} \beta_{n,i}^1 \left\|Su_n - T^i u_n\right\| + \sum_{i=1}^{q_2} \beta_{n,i}^1 \left\|T^i u_n - T^i x_n^1\right\| \\ &\leq \beta_{n,0}^1 \left\|Su_n - Sx_n^2\right\| + \sum_{i=1}^{q_2} \beta_{n,i}^1 \left\|Su_n - T^i u_n\right\| \\ &\quad + \sum_{i=1}^{q_2} \beta_{n,i}^1 \delta^i \left\|Su_n - Sx_n^1\right\| + \sum_{i=1}^{q_2} \beta_{n,i}^1 \sum_{j=0}^{i} \binom{i}{j} \delta^{i-j}\varphi^j \left\|Su_n - T^i u_n\right\|.\end{aligned} \tag{15}$$

Following the method of proof in (13), we can write

$$\|Su_n - T^i u_n\| \leq (1+\delta^i)\|Su_n - p\|. \tag{16}$$

Substituting (16) in (15) and simplifying, we obtain

$$\begin{aligned}\left\|Su_n - Sx_n^1\right\| \leq & \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i}\left\|Su_n - Sx_n^2\right\| \\ & + \frac{\sum_{i=1}^{q_2}\beta_{n,i}^1\left(1+\delta^i\right)}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i}\|Su_n - p\| \\ & + \frac{\sum_{i=1}^{q_2}\beta_{n,i}^1\sum_{j=0}^{i}\binom{i}{j})\delta^{i-j}\varphi^j\left(1+\delta^i\right)\|Su_n - p\|}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i}.\end{aligned} \tag{17}$$

Also, using (4), (7), and (10), we get

$$\begin{aligned}\left\|Su_n - Sx_n^2\right\| \leq & \frac{\beta_{n,0}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i}\left\|Su_n - Sx_n^3\right\| \\ & + \frac{\sum_{i=1}^{q_3}\beta_{n,i}^2\left(1+\delta^i\right)}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i}\|Su_n - p\| \\ & + \frac{\sum_{i=1}^{q_3}\beta_{n,i}^2\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left(1+\delta^i\right)\|Su_n - p\|}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i}.\end{aligned} \tag{18}$$

Substituting (17) and (18) in (14), we get

$$\begin{aligned}\|Su_{n+1} - Sx_{n+1}\| \leq & \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i} \\ & \times\frac{\beta_{n,0}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i}\left\|Su_n - Sx_n^3\right\| \\ & + \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i}\frac{\sum_{i=1}^{q_3}\beta_{n,i}^2(1+\delta^i)}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i}\|Su_n - p\| \\ & + \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i} \\ & \times\frac{\sum_{i=1}^{q_3}\beta_{n,i}^2\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left((1+\delta^i)\|Su_n - p\|\right)}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i} \\ & + \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\frac{\sum_{i=1}^{q_2}\beta_{n,i}^1(1+\delta^i)}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i}\|Su_n - p\| \\ & + \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\frac{\sum_{i=1}^{q_2}\beta_{n,i}^1\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left((1+\delta^i)\|Su_n - p\|\right)}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{(1)}\delta^i}\end{aligned} \tag{19}$$

$$+\frac{\sum_{i=1}^{q_1}\alpha_{n,i}\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left((1+\delta^i)\|Su_{n+1}-p\|\right)}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}.$$

Continuing this process to $k-1$ and simplifying, we have

$$\begin{aligned}\|Su_{n+1}-Sx_{n+1}\| &\leq \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\,\frac{\beta^1_{n,0}}{1-\sum_{i=1}^{q_2}\beta^1_{n,i}\delta^i}\,\frac{\beta^2_{n,0}}{1-\sum_{i=1}^{q_3}\beta^2_{n,i}\delta^i}\\
&\quad\cdots\frac{\beta^{k-2}_{n,0}}{1-\sum_{i=1}^{q_{k-1}}\beta^{k-2}_{n,i}\delta^i}\,\frac{\beta^{k-1}_{n,0}}{1-\sum_{i=1}^{q_k}\beta^{k-1}_{n,i}\delta^i}\|Su_n-Sx_n\|\\
&\quad+\Bigg[\frac{\beta^1_{n,0}}{1-\sum_{i=1}^{q_2}\beta^{(1)}_{n,i}\delta^i}\,\frac{\beta^2_{n,0}}{1-\sum_{i=1}^{q_3}\beta^2_{n,i}\delta^i}\\
&\quad\cdots\frac{\beta^{k-2}_{n,0}}{1-\sum_{i=1}^{q_{k-1}}\beta^{k-2}_{n,i}\delta^i}\,\frac{\sum_{i=1}^{q_k}\beta^{k-1}_{n,i}(1+\delta^i)}{1-\sum_{i=1}^{q_k}\beta^{k-1}_{n,i}\delta^i}\\
&\quad+\frac{\beta^{(1)}_{n,0}}{1-\sum_{i=1}^{q_2}\beta^1_{n,i}\delta^i}\cdots\frac{\beta^{k-2}_{n,0}}{1-\sum_{i=1}^{q_{k-1}}\beta^{k-2}_{n,i}\delta^i}\,\frac{\sum_{i=1}^{q_{k-1}}\beta^{k-2}_{n,i}(1+\delta^i)}{1-\sum_{i=1}^{q_{k-1}}\beta^{k-2}_{n,i}\delta^i}\\
&\quad+\frac{\sum_{i=1}^{q_2}\beta^1_{n,i}(1+\delta^i)}{1-\sum_{i=1}^{q_2}\beta^1_{n,i}\delta^i}\Bigg]\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\|Su_n-p\|\\
&\quad+\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\,\frac{\beta^1_{n,0}}{1-\sum_{i=1}^{q_2}\beta^1_{n,i}\delta^i}\,\frac{\beta^2_{n,0}}{1-\sum_{i=1}^{q_3}\beta^2_{n,i}\delta^i}\\
&\quad\cdots\frac{\beta^{k-2}_{n,0}}{1-\sum_{i=1}^{q_{k-1}}\beta^{k-2}_{n,i}\delta^i}\,\frac{\sum_{i=1}^{q_k}\beta^{k-1}_{n,i}\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j(1+\delta^i)\|Su_n-p\|}{1-\sum_{i=1}^{q_k}\beta^{k-1}_{n,i}\delta^i}\\
&\quad+\frac{\sum_{i=1}^{q_1}\alpha_{n,i}\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j(1+\delta^i)\|Su_{n+1}-p\|}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}.\end{aligned}\tag{20}$$

Recall that

$$\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\leq\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i+\alpha_{n,0}.$$

Let $\delta^i\leq\delta<1$, then

$$\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i+\alpha_{n,0}\leq[(1-\alpha_{n,0})\delta+\alpha_{n,0}].\tag{21}$$

Putting (21) in (20), we get

$$\|Su_{n+1}-Sx_{n+1}\|\leq[1-\lambda_n]\|Su_n-Sx_n\|+e_n,\tag{22}$$

where $\lambda_n=(1-\alpha_{n,0})(1-\delta)$ and

$$e_n=\left\{\left[(1-\beta^1_{n,0})\delta+\beta^1_{n,0}\right]\left[(1-\beta^2_{n,0})\delta+\beta^2_{n,0}\right]\right.$$

$$\ldots\left[(1-\beta_{n,0}^{k-2})\delta+\beta_{n,0}^{k-2}\right]\frac{\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i}$$
$$\left[(1-\beta_{n,0}^{1})\delta+\beta_{n,0}^{1}\right]\ldots\frac{\sum_{i=1}^{q_{k-1}}\beta_{n,i}^{k-2}}{1-\sum_{i=1}^{q_{k-1}}\beta_{n,i}^{k-2}\delta^i}$$
$$\left.+\frac{\sum_{i=1}^{q_2}\beta_{n,i}^{1}}{1-\sum_{i=1}^{q_2}\beta_{n,i}^{1}\delta^1}\right\}[(1-\alpha_{n,0})\delta+\alpha_{n,0}](1+\delta^i)||Su_n-p||$$
$$+[(1-\alpha_{n,0})\delta+\alpha_{n,0}][(1-\beta_{n,0}^{1})\delta+\beta_{n,0}^{1}]\ldots[(1-\beta_{n,0}^{k-2})\delta+\beta_{n,0}^{k-2}]$$
$$\times\frac{\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i}\varphi^j(1+\delta^i)\|Su_n-p\|.$$

Applying Lemma 1.12 in (22), it follows that $\lim_{n\to\infty}\|Su_n-Sx_n\|=0$. Since by assumption $\lim_{n\to\infty}Su_n=p$, we have that $\lim_{n\to\infty}Sx_n=p$.

(ii)→(i). If $\lim_{n\to\infty}Sx_n=p$, then using (4), (7), and (10), we have

$$\begin{aligned}\|Sx_{n+1}-Su_{n+1}\| &\le \alpha_{n,0}\left\|Sx_n^1-Su_n\right\|+\sum_{i=1}^{q_1}\alpha_{n,i}\left\|T^ix_{n+1}-T^iu_{n+1}\right\|\\ &\le \alpha_{n,0}\left\|Sx_n^1-Su_n\right\|+\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i\|Sx_{n+1}-Su_{n+1}\|\\ &\quad+\sum_{i=1}^{q_1}\alpha_{n,i}\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left\|Sx_{n+1}-T^ix_{n+1}\right\|.\end{aligned}\tag{23}$$

From (23), it follows that

$$\begin{aligned}\left\|Sx_{n+1}-T^ix_{n+1}\right\| &= \left\|Sx_{n+1}-Sp+Tp-T^ix_{n+1}\right\|\\ &\le \|Sx_{n+1}-p\|+\left\|Tp-T^ix_{n+1}\right\|\\ &\le \|Sx_{n+1}-p\|+\delta^i\|p-Sx_{n+1}\|\\ &\quad+(\sum_{i=1}^{q_1}\alpha_{n,i})(\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j(\|Sp-Tp\|))\\ &= (1+\delta^i)\|Sx_{n+1}-p\|.\end{aligned}\tag{24}$$

Substituting (24) in (23) and simplifying, we obtain

$$\begin{aligned}\|Sx_{n+1}-Su_{n+1}\| &\le \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\left\|Sx_n^1-Su_n\right\|\\ &\quad+\frac{\sum_{i=1}^{q_1}\alpha_{n,i}\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j(1+\delta^i)\|Sx_{n+1}-p\|}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}.\end{aligned}\tag{25}$$

Using (4), (7), and (11), we have

$$\begin{aligned}\left\|Sx_n^1 - Su_n\right\| &\le \beta_{n,0}^1 \left\|Sx_n^2 - Su_n\right\| + \sum_{i=1}^{q_2} \beta_{n,i}^1 \left\|T^i x_n^1 - Sx_n^1\right\| \\ &\quad + \sum_{i=1}^{q_2} \beta_{n,i}^1 \left\|Sx_n^1 - Su_n\right\| \\ &\le \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2} \beta_{n,i}^1} \left\|Sx_n^2 - Su_n\right\| \\ &\quad + \frac{\sum_{i=1}^{q_2} \beta_{n,i}^1}{1-\sum_{i=1}^{q_2} \beta_{n,i}^1} \left\|T^i x_n^1 - Sx_n^1\right\|.\end{aligned} \tag{26}$$

From (26), we get

$$\begin{aligned}\left\|Sx_n^1 - T^i x_n^1\right\| &= \left\|Sx_n^1 - Sp + Tp - T^i x_n^1\right\| \\ &\le \left\|Sx_n^1 - p\right\| + \left\|Tp - T^i x_n^1\right\| \\ &\le \left\|Sx_n^1 - p\right\| + \delta^i \left\|Sx_n^1 - p\right\| + \sum_{j=0}^{i} \binom{i}{j} \delta^{i-j} \varphi^j \|Sp - Tp\| \\ &= \left(1+\delta^i\right) \left\|Sx_n^1 - p\right\|,\end{aligned} \tag{27}$$

$$\begin{aligned}\left\|Sx_n^1 - p\right\| &\le \beta_{n,0}^1 \left\|Sx_n^2 - p\right\| + \sum_{i=1}^{q_2} \beta_{n,i}^1 \left\|T^i x_n^1 - p\right\| \\ &\le \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2} \beta_{n,i}^1 \delta^i} \left\|Sx_n^2 - p\right\|,\end{aligned} \tag{28}$$

$$\begin{aligned}\left\|Sx_n^2 - p\right\| \le & \frac{\beta_{n,0}^2}{1-\sum_{i=1}^{q_3} \beta_{n,i}^2 \delta^i} \frac{\beta_{n,0}^3}{1-\sum_{i=1}^{q_4} \beta_{n,i}^3 \delta^i} \\ & \cdots \frac{\beta_{n,0}^{k-2}}{1-\sum_{i=1}^{q_{k-1}} \beta_{n,i}^{k-2} \delta^i} \frac{\beta_{n,0}^{k-1}}{1-\sum_{i=1}^{q_k} \beta_{n,i}^{k-1} \delta^i} \left\|Sx_n - p\right\|,\end{aligned} \tag{29}$$

and

$$\left\|Sx_n^{k-1} - p\right\| \le \frac{\beta_{n,0}^{k-1}}{1-\sum_{i=1}^{q_k} \beta_{n,i}^{k-1} \delta^i} \|Sx_n - p\|. \tag{30}$$

Substituting (28), (29) and (30) in (27) yields

$$\begin{aligned}\left\|Sx_n^1 - T^i x_n^1\right\| \le & \left(1+\delta^i\right) \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2} \beta_{n,i}^1 \delta^i} \frac{\beta_{n,0}^2}{1-\sum_{i=1}^{q_3} \beta_{n,i}^2 \delta^i} \\ & \cdots \frac{\beta_{n,0}^{k-2}}{1-\sum_{i=1}^{q_{k-1}} \beta_{n,i}^{k-2} \delta^i} \frac{\beta_{n,0}^{k-1}}{1-\sum_{i=1}^{q_k} \beta_{n,i}^{k-1} \delta^i} \left\|Sx_n - p\right\|.\end{aligned} \tag{31}$$

Substituting (31) in (26), we obtain

$$
\begin{aligned}
\left\|Sx_n^1 - Su_n\right\| \leq \; & \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1} \; \left\|Sx_n^2 - Su_n\right\| \\
& + \left(1+\delta^i\right) \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i} \frac{\beta_{n,0}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i} \\
& \cdots \frac{\beta_{n,0}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i} \frac{\sum_{i=1}^{q_2}\beta_{n,i}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1} \left\|Sx_n - p\right\|.
\end{aligned} \tag{32}
$$

The estimates for $\|Sx_n^2 - Su_n\|, \ldots, \|Sx_n^{k-2} - Su_n\|$ and $\|Sx_n^{k-1} - Su_n\|$ are similarly obtained and substituted in (32) to get

$$
\begin{aligned}
\|Sx_n^1 - Su_n\| \leq \; & \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i} \frac{\beta_{n,0}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i} \\
& \cdots \frac{\beta_{n,0}^{k-2}}{1-\sum_{i=1}^{q_{k-1}}\beta_{n,i}^{k-2}\delta^i} \frac{\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i} \|Sx_n - Su_n\| \\
& + \left[\frac{\sum_{i=1}^{q_2}\beta_{n,i}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1} + \frac{\sum_{i=1}^{q_3}\beta_{n,i}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i} + \cdots \right. \\
& \quad \left. + \frac{\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i}\right] \\
& \times \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i} \frac{\beta_{n,0}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i} \\
& \cdots \frac{\beta_{n,0}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i} \left(1+\delta^i\right) \|Sx_n - p\|.
\end{aligned} \tag{33}
$$

Substituting (33) in (25), we have

$$
\begin{aligned}
\|Sx_{n+1} - Su_{n+1}\| \leq \; & \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i} \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i} \frac{\beta_{n,0}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i} \\
& \cdots \frac{\beta_{n,0}^{k-2}}{1-\sum_{i=1}^{q_{k-1}}\beta_{n,i}^{k-2}\delta^i} \frac{\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i} \|Sx_n - Su_n\| \\
& + \left[\frac{\sum_{i=1}^{q_2}\beta_{n,i}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1} + \frac{\sum_{i=1}^{q_3}\beta_{n,i}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i} + \cdots + \frac{\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i}\right] \\
& \times \frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i} \frac{\beta_{n,0}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1\delta^i} \frac{\beta_{n,0}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i}
\end{aligned} \tag{34}
$$

$$\cdots \frac{\beta_{n,0}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i}\left(1+\delta^i\right)\|Sx_n-p\|$$
$$+\frac{\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j(1+\delta^i)\|Sx_{n+1}-p\|}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}.$$

We know that

$$\frac{\alpha_{n,0}}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}\le\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i+\alpha_{n,0}.$$

Let $\delta^i\le\delta<1$, then

$$\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i+\alpha_{n,0}\le\left[(1-\alpha_{n,0})\delta+\alpha_{n,0}\right]. \tag{35}$$

Using (35) in (34), we have

$$\|Sx_{n+1}-Su_{n+1}\|\le[1-\lambda_n]\|Su_n-Sx_n\|+e_n, \tag{36}$$

where $\lambda_n=(1-\alpha_{n,0})(1-\delta)$ and

$$e_n=(1+\delta^i)\left[(1-\alpha_{n,0})\delta+\alpha_{n,0}\right]\left[(1-\beta_{n,0}^1)\delta+\beta_{n,0}^1\right]\left[(1-\beta_{n,0}^2)\delta+\beta_{n,0}^2\right]$$
$$\cdots\left[(1-\beta_{n,0}^{k-2})\delta+\beta_{n,0}^{k-2}\right]\left[\frac{\sum_{i=1}^{q_2}\beta_{n,i}^1}{1-\sum_{i=1}^{q_2}\beta_{n,i}^1}+\frac{\sum_{i=1}^{q_3}\beta_{n,i}^2}{1-\sum_{i=1}^{q_3}\beta_{n,i}^2\delta^i}+\cdots\right.$$
$$\left.+\frac{\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}}{1-\sum_{i=1}^{q_k}\beta_{n,i}^{k-1}\delta^i}\right]\|Sx_n-p\|$$
$$+\frac{\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i\sum_{j=0}^{i}\binom{i}{j}\delta^{i-j}\varphi^j\left(1+\delta^i\right)\|Sx_{n+1}-p\|}{1-\sum_{i=1}^{q_1}\alpha_{n,i}\delta^i}.$$

Using Lemma 1.12 in (36), it follows that $\lim_{n\to\infty}\|Sx_n-Su_n\|=0$. Since, by assumption, $\lim_{n\to\infty}Sx_n=p$, this implies that $\lim_{n\to\infty}Su_n=p$. □

Since the implicit Jungck–Kirk-multistep iterative scheme (7) generalizes other implicit Jungck–Kirk-type schemes (4), (5), and (6), Theorem 2.1 leads to the following corollary.

Corollary 2.2. *Let E, S, T, and p be the same as in Theorem 2.1. If $u_0=z_0=y_0=x_0\in E$, then the following statements are equivalent:*

(i) *Implicit Jungck–Kirk–Mann iteration* (4) *converges strongly to p;*
(ii) *Implicit Jungck–Kirk–Ishikawa iteration* (5) *converges strongly to p;*
(iii) *Implicit Jungck–Kirk–Noor iteration* (6) *converges strongly to p;*
(iv) *Implicit Jungck–Kirk-multistep iteration* (7) *converges strongly to p.*

Example 2.3. Consider the equation $f(x)=0$, where f is the real function defined on interval $[0,\pi/2]$ by $f(x)=x^2-(\pi/2)^2\cos x$. The function f can be decomposed as $f=\pi/2(S-T)$, where the maps S and T are the self mappings in $[0, \frac{\pi}{2}]$ defined by $S(x) = (2/\pi)x^2$ and $T(x) = (\pi/2)\cos x$. Clearly, S and T satisfy the contractive condition (11). They coincide at $\omega \approx 1.0792$ and we have $p = S\omega = T\omega \approx 0.7415$. Thus, ω is a solution to $f(x) = 0$. However, if S and T are weakly compatible, then $p = Sp = Tp \approx 0.7415$.

From Theorem 2.1 and Corollary 2.2, the implicit Jungck–Kirk multistep, implicit Jungck–Kirk–Noor, implicit Jungck–Kirk–Ishikawa, and implicit Jungck–Kirk–Mann hybrid iterations given, respectively, in (7), (6), (5), and (4) converge to p. Using MATLAB, we have the following result.

n	x_n	y_n	z_n	u_n	Sx_n	Sy_n	Sz_n	Su_n
0	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
1	1.0472	1.0468	1.0462	1.0459	0.6982	0.6926	0.6912	0.6908
2	1.0739	1.0711	1.0692	1.0673	0.7343	0.7337	0.7332	0.7328
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5	1.0791	1.0784	1.0741	1.0712	0.7412	0.7408	0.7405	0.7402
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
7	1.0792	1.0791	1.0767	1.0762	0.7415	0.7412	0.7410	0.7407
8	1.0792	1.0792	1.0791	1.0786	0.7415	0.7415	0.7412	0.7410
9	1.0792	1.0792	1.0792	1.0791	0.7415	0.7415	0.7415	0.7412
10	1.0792	1.0792	1.0792	1.0792	0.7415	0.7415	0.7415	0.7415

3. Conclusion

In the present work, we have investigated the equivalence of various implicit Jungck-Kirk-type iterative schemes via weakly compatible generalized contractive-like operators in normed linear spaces. We also gave an example to justify the equivalence results.

Competing interest. The authors declare that there is no competing interest.

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