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# On Lucas-balancing zeta function

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Abstract. In the present study a new modification of Riemann zeta function known as Lucas-balancing zeta function is introduced. The Lucas-balancing zeta function admits its analytic continuation over the whole complex plane except its poles. This series converges to a fixed rational number $-\frac{1}{2}$ at negative odd integers. Further, in accordance to Dirichlet L-function, the analytic continuation of Lucas-balancing L-function is also discussed.

## 1. Introduction

It is well known that all the functions defined by a series $\sum_{n=1}^{\infty} a_n n^{-s}$, where $a_n$ is a complex sequence with $s = \sigma + it \in \mathbb{C}$, are called the Dirichlet series. Such series are mostly related to each other in their algebraic properties in terms of analyticity when $a_n$ is a multiplicative function. As usual, for $a_n = 1$ and $\Re(s) > 1$, the series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is known as the Riemann zeta function (see [2, 4]). It is a prototypical Dirichlet series which converges absolutely to an analytic function for all real values of $s$ greater than 1 and diverges for all the other values of $s$. Riemann [2] proved that the zeta function defined by the series on the half plane is analytically continued to all the complex values except $s = 1$, for which it gives a harmonic series converging to $+\infty$. So Riemann zeta function is holomorphic on the whole complex plane except for the simple pole at $s = 1$, and the residue is given by 1. The functional equation of Riemann zeta function shows that it has zeros at $-2, -4, -6, \dots$, which are the trivial zeros, and has rational values at negative odd integers, which are known as the Bernoulli numbers

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(see [12]). Any non-trivial zeros lies on the open strip $s \in \mathbb{C},\ 0 < \Re(s) < 1$, which is known as the critical strip.

Fibonacci zeta function is defined by

$$\zeta_F(s) = \sum_{n\in\mathbb{N}} F_n^{-s},$$

where $F_n$ denotes the $n$-th Fibonacci number (see [10]). Since the $n$-th Fibonacci number shows exponential growth, it can be easily shown that such series converges for $\Re(s) > 0$, however it has been proved by André-Jeannin [1] that such series is an irrational number. Its analytic continuation can derived easily by using the similar methods used for Riemann zeta function and Hurwitz zeta function. It was proved by Navas [8] (based on its Binet formula) and Egami [5] that such function is analytically continued in the whole complex plane except for the simple poles. Elsner et al. [6] obtained that the values of $\zeta_F$ at positive even integers like $2, 4, 6, \dots$ are algebraically independent and also the function is transcendental at those points however its values at positive odd integers are still unknown. Kamano [7] considered the Lucas zeta function and L-function, which are generalizations of Fibonacci zeta function and L-function, respectively, and studied their analytic continuation.

In [11] Rout and Panda have considered another generalization of Dirichlet series given by

$$\zeta_B(s) = \sum_{n=1}^{\infty} B_n^{-s},\ \ \Re(s) > 1,\ \ s \in \mathbb{C},$$

known as balancing Dirichlet series. They showed that $\zeta_B(s)$ is meromorphically continued all over the complex plane $\mathbb{C}$. This inspired us to consider about the analytic continuation of a new series

$$\zeta_C(s) = \sum_{n=1}^{\infty} C_n^{-s},$$

where $C_n$ is the $n$-th Lucas-balancing number (see [9]). We call the series the Lucas-balancing zeta function. It is worthy to define balancing and Lucas-balancing numbers. Balancing numbers are obtained from a simple Diophantine equation and satisfy the recurrence relation $B_{n+1} = 6B_n - B_{n-1}$, $n \geq 1$, with initial values $B_0 = 0$ and $B_1 = 1$. Lucas-balancing numbers satisfy the same recurrence relation with different initials $C_0 = 1$ and $C_1 = 3$ (see [3, 9]). The Binet formulas for both these numbers are given by

$$B_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2},\quad C_n = \frac{\lambda_1^n + \lambda_2^n}{2},$$

where $\lambda_1 = 3 + 2\sqrt{2}$ and $\lambda_2 = 3 - 2\sqrt{2}$.

In this paper, we will discuss about the analytic continuation of Lucas-balancing zeta function and its values at integral arguments which are described below. Also it will be verified that such function has simple poles at negative even integers however it has no trivial zeros.

## 2. Analytic continuation of $\zeta_C(s)$ over $\mathbb{C}$

It is well known that analytic continuation is the technique of extending the domain of given analytic function and often helps in defining the values of a function in the region. In this section, we shall discuss about the extension of Lucas-balancing zeta function on $\mathbb{C}$.

**Theorem 2.1.** *The function $\zeta_C(s)$ can be meromorphically continued to the whole complex plane and can be expressed as*

$$\zeta_C(s) = 2^s \sum_{k=o}^{\infty} (-1)^k \binom{-s}{k} (\lambda_1^{s+2k} - 1)^{-1}.$$

*It is holomorphic except at the simple poles*

$$s^* = -2k + \frac{2\pi i n}{\log \lambda_1}, \quad k \geq 0, \tag{2.1}$$

*and the residue at $s = s^*$ is given by*

$$\mathrm{Res}\left[\zeta_C(s) : s^*\right] = \frac{2^{s^*}(-1)^k \binom{-s^*}{k}}{\log \lambda_1}.$$

*Proof.* The recurrence relation for Lucas-balancing number is

$$C_{n+1} = 6C_n - C_{n-1}, \quad C_0 = 1, \; C_1 = 3,$$

and its Binet formula is given by

$$C_n = \frac{\lambda_1^n + \lambda_2^n}{2}.$$

For any complex number $z$,

$$\begin{aligned} C_n^z &= \left(\frac{\lambda_1^n + \lambda_2^n}{2}\right)^z = 2^{-z}\left(\lambda_1^n + \lambda_2^n\right)^z \\ &= 2^{-z}\lambda_1^{nz}\left(1 + \left(\frac{\lambda_2}{\lambda_1}\right)^n\right)^z = 2^{-z}\lambda_1^{nz}\left(1 + \left(\frac{1}{\lambda_1}\right)^{2n}\right)^z \\ &= 2^{-z}\lambda_1^{nz}\sum_{k=0}^{\infty}(-1)^k\binom{z}{k}\lambda_1^{-2nk}. \end{aligned}$$

This series expansion is only valid for all $z \in \mathbb{C}$ and $\lambda_1 > 0$. Now, substituting in the above expression with $z = -s$ , we have

$$C_n^{-s} = 2^s \lambda_1^{-ns} \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \lambda_1^{-2nk}.$$

Thus

$$\sum_{n=1}^{\infty} C_n^{-s} = 2^s \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \lambda_1^{-n(s+2k)}.$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} |C_n^{-s}| &= 2^s \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left| (-1)^k \binom{-s}{k} \lambda_1^{-n(s+2k)} \right| \\
&= 2^s \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left| \binom{-|s|}{k} \lambda_1^{-n(s+2k)} \right| \leq 2^s \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{-|s|}{k} \lambda_1^{-n(\sigma+2k)} \\
&= 2^s \sum_{n=1}^{\infty} \lambda_1^{-n\sigma} \sum_{k=0}^{\infty} (-1)^k \binom{-|s|}{k} \lambda_1^{-2nk} = 2^s \sum_{n=1}^{\infty} \lambda_1^{-n\sigma} (1 - \lambda_1^{-2n})^{-|s|} \\
&\leq 2^s (1 - \lambda_1^{-2})^{-|s|} \sum_{n=1}^{\infty} \lambda_1^{-n\sigma} < \infty.
\end{aligned}$$

Thus the power series representation of $\zeta_C(s)$ converges absolutely and hence it also converges, and can be analytic in $\mathbb{C}$. Now

$$\begin{aligned}
\zeta_C(s) = \sum_{n=1}^{\infty} C_n^{-s} &= 2^s \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \lambda_1^{-n(s+2k)} \\
&= 2^s \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \sum_{n=1}^{\infty} \left( \lambda_1^{-(s+2k)} \right)^n \\
&= 2^s \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \left( \frac{1}{1 - \lambda_1^{-s-2k}} - 1 \right) \\
&= 2^s \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \left( \frac{\lambda_1^{-s-2k}}{1 - \lambda_1^{-s-2k}} \right) = 2^s \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \left( \frac{1}{\lambda_1^{s+2k} - 1} \right) \\
&= 2^s \sum_{k=0}^{\infty} f_k(s),
\end{aligned}$$

where

$$f_k(s) = (-1)^k \binom{-s}{k} \left( \frac{1}{\lambda_1^{s+2k} - 1} \right).$$

Here $f_k(s)$ is an analytic function on $\mathbb{C}$ except at the poles which are derived by equating $\lambda_1^{s+2k} - 1 = 0$. This gives the values (2.1) of $s$. Now, the residue of $\zeta_C(s)$ at $s^*$ is given by

$$\mathrm{Res}\,[\zeta_C(s) : s^*] = \lim_{s \to s^*} (s - s^*)\zeta_C(s) = 2^{s^*}(-1)^k \binom{-s^*}{k} \frac{1}{\log \lambda_1}.$$

Thus the proof is completed. □

## 3. Values of $\zeta_C(s)$ at integral arguments

**3.1. Values at negative integers.** Since the poles of the function $\zeta_C(s)$ are determined by (2.1), the poles $s^*$ lie on the line $\Re(s) = -2k$ and are equally spaced in the interval of $\frac{2\pi i}{\log \lambda_1}$. So the even negative integers $-2, -4, -6, ...$ are its poles. Now we shall find out the values of $\zeta_C(s)$ at the negative odd integers. If $m$ is an odd natural number, then

$$\zeta_C(-m) = 2^{-m} \sum_{k=0}^{\infty} (-1)^k \binom{m}{k} \frac{1}{\lambda_1^{-m+2k} - 1}, \quad \binom{m}{k} = 0,\ k > m.$$

Let $\sigma_k = (-1)^k \binom{m}{k} \frac{1}{(\lambda_1^{-m+2k} - 1)}$ and $\alpha_k = \sigma_k - \sigma_{m-k}$. One has

$$\begin{aligned}
\alpha_k &= (-1)^k \binom{m}{k} \frac{1}{\lambda_1^{-m+2k} - 1} - (-1)^{m-k} \binom{m}{m-k} \frac{1}{\lambda_1^{m-2k} - 1} \\
&= (-1)^k \binom{m}{k} \left[ \frac{1}{\lambda_1^{-m+2k} - 1} + \frac{1}{\lambda_1^{m-2k} - 1} \right] \\
&= (-1)^k \binom{m}{k} \left[ \frac{1}{\lambda_1^{-m+2k} - 1} + \frac{1}{\lambda_2^{-m+2k} - 1} \right] \\
&= (-1)^k \binom{m}{k} \left[ \frac{\lambda_1^{-m+2k} - 1 + \lambda_2^{-m+2k} - 1}{(\lambda_1^{-m+2k} - 1)(\lambda_2^{-m+2k} - 1)} \right] = (-1)^{k+1} \binom{m}{k}.
\end{aligned}$$

So $\alpha_{m-k} = (-1)^{m-\ k+1} \binom{m}{m-k} = -\alpha_k$. That means $\alpha_k + \alpha_{m-k} = 0$. Now,

$$\begin{aligned}
\zeta_C(-m) &= 2^{-m} \sum_{k=0}^{\infty} \sigma_k = 2^{-m} \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k \alpha_k \\
&= 2^{-m-1} \sum_{k=0}^{\infty} (-1)^k (-1)^{k+1} \binom{m}{k} = -2^{-m-1} \sum_{k=0}^{\infty} \binom{m}{k} \\
&= -2^{-m-1+m} = -\frac{1}{2},
\end{aligned}$$

which is a rational number.

**3.2. Values at positive integers.**

**Theorem 3.1.** *The values of Lucas-balancing zeta function at positive integers $m$ are given by*

$$\zeta_C(m) = 2^m \sum_{l=1}^{\infty} a_l \lambda_1^{-l}, \quad a_l = \sum_{d|l,\, d\in S_m} \binom{\frac{d+m-2}{2}}{m-1}.$$

*Proof.* We know that

$$C_n^s = 2^{-s} \sum_{k=0}^{\infty} (-1)^k \binom{s}{k} \lambda_1^{n(s-2k)} \text{ and } (-1)^k \binom{-s}{k} = \binom{s+k-1}{k}.$$

So, for all $m \in \mathbb{N}$,

$$\begin{aligned} C_n^{-m} &= 2^m \sum_{k=0}^{\infty} (-1)^k \binom{-m}{k} \lambda_1^{-n(m+2k)} = 2^m \sum_{k=0}^{\infty} \binom{m+k-1}{k} \lambda_1^{-n(m+2k)} \\ &= 2^m \sum_{k=0}^{\infty} \binom{m+k-1}{m-1} \lambda_1^{-n(m+2k)}. \end{aligned}$$

Taking $d = m + 2k$ and $S_m = \{d \geq m : d \equiv m\,(mod\ 2)\}$, we have

$$C_n^{-m} = 2^m \sum_{d\in S_m} \binom{\frac{d+m-2}{2}}{m-1} \lambda_1^{-nd}.$$

To find out the sum of the above expression over $n$ , collecting the like powers $l = nd$ and restricting $d$ as $d|l$ whenever $l$ runs over natural numbers, we can have the previous expression as

$$\sum_{n=1}^{\infty} C_n^{-m} = 2^m \sum_{l=1}^{\infty} \sum_{d|l,\, d\in S_m} \binom{\frac{d+m-2}{2}}{m-1} \lambda_1^{-l} = 2^m \sum_{l=1}^{\infty} a_l \lambda_1^{-l}.$$

This completes the proof. □

In particular, when $m = 1$, we have $S_1 = \{d \geq 1 : d \equiv 1\ (mod\,2\,)\}$. Then

$$\sum_{n=1}^{\infty} C_n^{-1} = 2 \sum_{l=1}^{\infty} \sum_{d|l} \binom{\frac{d-1}{2}}{0} \lambda_1^{-l} = 2 \sum_{l=1}^{\infty} \sum_{d|l,\, d\in S_1} 1 \lambda_1^{-l} = 2 \sum_{l=1}^{\infty} d_1(l) \lambda_1^{-l},$$

where

$$d_1(l) = \sum_{d|l,\, d\equiv 1 (mod\, 2)} 1.$$

Thus we have

$$\sum_{n=1}^{\infty} C_{2n}^{-1} = 2 \sum_{l\equiv 0 (mod\, 2)} d_1(l) \lambda_1^{-l} \text{ and } \sum_{n=1}^{\infty} C_{2n-1}^{-1} = 2 \sum_{l\equiv 1 (mod\, 2)} d_1(l) \lambda_1^{-l}.$$

## 4. The Lucas-balancing L-function

The Dirichlet L-function is the function of the form

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(\ n)\ n^{-s}.$$

This function was first introduced by Dirichlet in [2] in order to prove the prime number theorem in arithmetic progressions and hence it was named after him. He proved that $L(s,\chi)$ is non-zero at $s=1$. Moreover, the Dirichlet L-function attains its simple pole at $s=1$ if the corresponding $\chi$ is principal.

The zeros of $L(s,\chi)$ mainly depend upon $\chi$. If $\chi$ is a primitive character such that $\chi(-1)=1$, then the negative even integers are its only zeros and if $\chi(-1)=-1$, then the negative odd integers are the only zeros with $\Re(s)<0$. In [11], there is a function known as balancing L-function which is a new modification of Dirichlet L-function. It has been defined as

$$L_{\mathcal{B}}(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\mathcal{B}_n^{-s},\quad \Re(s)>1.$$

Now keeping an eye on balancing L-function, its definition helped us to define another function

$$L_C(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{C_n^s},\quad \Re(s)>1.$$

This function seems to be very similar to Dirichlet L-function and balancing L-function. Rout and Panda [11] have discussed about the meromorphic continuation of $L_{\mathcal{B}}(s,\chi)$. In this section, we will discuss about the analytic continuation of Lucas-balancing L-function.

We know that the Hurwitz zeta function $\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$ is a generalisation of $\zeta(s)$ and $L(s,\chi)$ (see [2]). Similarly we can write the Lucas-balancing L-function in terms of $\zeta_C(s)$ and we can define it as

$$\zeta_C(s,(r,p))=\sum_{n\geq 1,\, n\equiv r(mod\, p)}\frac{1}{C_n^s}.$$

If we take $r=p=1$, then $\zeta_C(s(1,1))=\zeta_C(s)$. Further,

$$\begin{aligned}\zeta_C(s,(r,p))&=\sum_{n=0}^{\infty}\frac{1}{C_{pn+r}^s}=2^s\sum_{n=0}^{\infty}\left(\lambda_1^{pn+r}+\lambda_2^{pn+r}\right)^{-s}\\&=2^s\sum_{n=0}^{\infty}\lambda_1^{-(pn+r)s}\left(1+\left(\frac{\lambda_2}{\lambda_1}\right)^{pn+r}\right)^{-s}\\&=2^s\sum_{n=0}^{\infty}\lambda_1^{-(pn+r)s}\sum_{k=0}^{\infty}(-1)^k\binom{-s}{k}\left(\frac{\lambda_2}{\lambda_1}\right)^{pn+r}\end{aligned}$$

$$= 2^s \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \lambda_1^{-(pn+r)(s+2k)}$$

$$= 2^s \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \frac{\lambda_1^{-(s+2k)r}}{1-\lambda_1^{-(s+2k)p}}.$$

This concludes that $\zeta_C(s,(r,p))$ is meromorphically continued over whole complex plane except at the poles

$$s_{k,n} = -2k + \frac{2\pi i n}{p \log \lambda_1}. \tag{4.1}$$

For the analytic continuation of Lucas-balancing L-function, we need to use another function, i.e., "Gauss sum" associated with $k$, which is defined as

$$G(n,\chi) = \sum_{m=1}^{k} \chi(m) \exp\left(\frac{2\pi i m n}{k}\right),$$

where $\chi$ is a Dirichlet character modulo $k$.

**Theorem 4.1.** *The function $L_C(s,\chi)$ is analytic over the whole complex plane except the poles* (4.1)*, and the residue at $s = s_{k,n}$ is given by*

$$\mathrm{Res}\left[L_C(s,\chi) : s_{k,n}\right] = 2^{s_{k,n}} (-1)^k \binom{-s_{k,n}}{k} \frac{1}{p \log \lambda_1} \chi(-1) G(n,\chi).$$

*Proof.* We know that

$$L_C(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{C_n^s} = \sum_{r=1}^{p} \chi(r) \zeta_C(s,(r,p)).$$

This clears that the poles of $L_C(s,\chi)$ are the same as the poles of $\zeta_C(s,(r,p))$. Thus likewise $\zeta_C(s,(r,p))$, $L_C(s,\chi)$ is also analytically continued all over complex plane except the poles. Again, in order to find the residue of $\zeta_C(s,(r,p))$ at the poles, we have

$$\mathrm{Res}\left[\zeta_C(s,(r,p)) : s_{k,n}\right] = 2^{s_{k,n}} (-1)^k \binom{-s_{k,n}}{k} \lim_{s \to s_{k,n}} (s - s_{k,n}) \frac{\lambda_1^{-(s+2k)r}}{1-\lambda_1^{-(s+2k)p}}$$

$$= 2^{s_{k,n}} (-1)^k \binom{-s_{k,n}}{k} \exp\left(\frac{-2\pi i n r}{p}\right) \lim_{s \to s_{k,n}} (s - s_{k,n}) \frac{1}{1-\lambda_1^{-(s+2k)p}}$$

$$= 2^{s_{k,n}} (-1)^k \binom{-s_{k,n}}{k} \exp\left(\frac{-2\pi i n r}{p}\right) \frac{1}{p \log \lambda_1}.$$

Hence the residue of $L_C(s,\chi)$ at poles is given by

$$\mathrm{Res}\left[L_C(s,\chi) : s_{k,n}\right] = \sum_{r=1}^{p} \chi(\ r) \mathrm{Res}\left[\zeta_C(s,(r,p)) : s_{k,n}\right]$$

$$= 2^{s_{k,n}}(-1)^k \binom{-s_{k,n}}{k} \frac{1}{p\log\lambda_1} \sum_{r=1}^{p} \chi(r) \exp\left(\frac{-2\pi i n r}{p}\right).$$

Using the definition of $G(n,\chi)$, we get

$$\begin{aligned} \operatorname{Res}\left[L_C(s,\chi) : s_{k,n}\right] &= 2^{s_{k,n}}(-1)^k \binom{-s_{k,n}}{k} (p\log\lambda_1)^{-1} G(-n,\chi) \\ &= 2^{s_{k,n}}(-1)^k \binom{-s_{k,n}}{k} (p\log\lambda_1)^{-1} \chi(-1) G(n,\chi). \end{aligned}$$

Hence this ends the proof of the theorem. □

**Theorem 4.2.** *Let us consider $\chi$ as a non-principal character modulo $p$. Then $L_C(-n,\chi) = 0$ for all $n \in \mathbb{N}$.*

*Proof.* Combining the definitions of $L_C(s,\chi)$ and $\zeta_C(s,(r,p))$, we get

$$L_C(s,\chi) = \sum_{r=1}^{p} \chi(r) 2^s \sum_{k=0}^{\infty} (-1)^k \binom{-s}{k} \frac{\lambda_1^{-(s+2k)r}}{1-\lambda_1^{-(s+2k)p}}.$$

When $s = -n$ , we have the above expression as

$$L_C(-n,\chi) = \sum_{r=1}^{p} \chi(r) 2^{-n} \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \frac{\lambda_1^{(n-2k)r}}{1-\lambda_1^{(n-2k)p}}.$$

When $n$ is even, the points $-2,-4,-6,\dots$ are the poles of $L_C(-n,\chi)$, and give its values as zero. Again, when $n$ is odd, let

$$\sigma_k = (-1)^k \binom{n}{k} \frac{\lambda_1^{(n-2k)r}}{1-\lambda_1^{(n-2k)p}} \quad \text{and} \quad \alpha_k = \sigma_k - \sigma_{n-k}.$$

Then

$$\begin{aligned} \alpha_k &= (-1)^k \binom{n}{k} \frac{\lambda_1^{(n-2k)r}}{1-\lambda_1^{(n-2k)p}} - (-1)^{n-k} \binom{n}{n-k} \frac{\lambda_1^{(-n+2k)r}}{1-\lambda_1^{(-n+2k)p}} \\ &= (-1)^k \binom{n}{k} \left[ \frac{\lambda_1^{(n-2k)r}}{1-\lambda_1^{(n-2k)p}} + \frac{\lambda_1^{(-n+2k)r}}{1-\lambda_1^{(-n+2k)p}} \right] \\ &= (-1)^k \binom{n}{k} \left[ \frac{\lambda_1^{(n-2k)r}}{1-\lambda_1^{(n-2k)p}} + \frac{\lambda_2^{(n-2k)r}}{1-\lambda_2^{(n-2k)p}} \right] \\ &= (-1)^k \binom{n}{k} (-1) = (-1)^{k+1} \binom{n}{k}. \end{aligned}$$

Now, $\alpha_{n-k} = (-1)^{n-k+1}\binom{n}{k} = -\alpha_k$, and hence

$$\sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \frac{\lambda_1^{(n-2k)r}}{1-\lambda_1^{(n-2k)p}} = \sum_{k=0}^{n} \sigma_k = \frac{1}{2} \sum_{k=0}^{n} (-1)^k \alpha_k$$

$$= \frac{1}{2}\sum_{k=0}^{n}(-1)^k(-1)^{k+1}\binom{n}{k} = -\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k} = -2^{n-1}.$$

Therefore,

$$L_C(-n,\chi) = -\frac{1}{2}\sum_{r=1}^{p}\chi(r) = 0 \text{ as } \sum_{r=1}^{p}\chi(r) = 0.$$

This completes the proof. □

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