ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 22, Number 1, June 2018
Available online at http://acutm.math.ut.ee

Remarks on locally closed set

Shyamapada Modak and Takashi Noiri

Abstract. In this paper, we study properties of pre-I-open sets and $\mathcal{C}_I$-continuity. We also establish a relation between LC-continuity and $\mathcal{C}_I$-continuity. Moreover, we investigate properties of pre^*_I-open sets and pre^*_I-continuity defined in [3] and [6], respectively.

1. Introduction and preliminaries

The concept of ideals [13] is well known in topological spaces. Let (X, τ) be a topological space and let $\wp(X)$ be the family of all subsets of X. A subfamily I of $\wp(X)$ is called an *ideal* if it satisfies the following properties:
(1) $A \in I$ and $B \in I$ implies $A \cup B \in I$;
(2) $A \in I$ and $B \subseteq A$ implies $B \in I$.
A topological space (X, τ) with an ideal I is called an *ideal topological space* and is denoted by (X, τ, I). A new topology τ^* for X, called the $*$-topology [12], has been constructed from (X, τ, I). It is known that τ^* is finer than τ; and the family $\{V \setminus J : V \in \tau, J \in I\}$ is a base of τ^*. The closure (respectively, interior) of a subset A of X in the topological space (X, τ^*) is denoted by $Cl^*(A)$ (respectively, $Int^*(A)$). Many mathematicians have been interested in (X, τ^*), and different types of generalized open and closed sets have been defined with the help of the closure and the interior operators with respect to τ and τ^*. Using these sets, different kinds of continuity have been defined and investigated.

In this paper, we investigate relations between pre-I-closed sets [2], $\mathcal{C}_I$-sets [5], locally closed sets [1], and A^*_I-sets [5]. We also study several properties of pre^*_I-open sets [3] and pre^*_I-continuity [6].

The following lemma plays an important role in the study of this paper.

Received February 1, 2017.
2010 *Mathematics Subject Classification.* 54A05; 54A10; 54B10.
Key words and phrases. Pre^*-I-open; pre-I-closed; locally closed set; $\mathcal{C}_I$-continuous.
http://dx.doi.org/10.12697/ACUTM.2018.22.06

Lemma 1.1 (see [12, 11, 16]). *Let (X, τ, I) be an ideal topological space. The following statements are equivalent:*

(1) $X = X^*$.
(2) $\tau \cap I = \{\emptyset\}$.
(3) *If* $J \in I$, *then* $Int(J) = \emptyset$.
(4) *For every* $U \in \tau$, $U \subseteq U^*$.
(5) *For every* $U \in \tau$, $U^* = Cl(U)$.
(6) *For every semiopen set* G *in* (X, τ), $G \subseteq G^*$.

2. Pre-I-open sets

Definition 2.1 (see [2]). A subset P of an ideal topological space (X, τ, I) is called pre-I-open if $P \subseteq Int(Cl^*(P))$.

The collection of all pre-I-open sets in (X, τ, I) is denoted by $PIO(X)$.

Definition 2.2 (see [14]). A subset P of a topological space (X, τ) is called preopen if $P \subseteq Int(Cl(P))$.

The collection of all preopen sets in (X, τ) is denoted by $PO(X)$.

If a set is pre-I-open, then it is obviously preopen. For the converse, we have following proposition.

Proposition 2.3 (see [15]). *Let (X, τ, I) be an ideal topological space, where I is codense. Then $PIO(X) = PO(X, \tau^*(I))$, where $PO(X, \tau^*(I))$ denotes the collection of all preopen sets in (X, τ^*).*

If $I = \{\emptyset\}$ for an ideal topological space (X, τ, I), then $\tau \cap I = \{\emptyset\}$ and hence $PIO(X) = PO(X, \tau^*(I))$.

Let I_n denote the collection of all nowhere dense subsets of the topological space (X, τ). Then I_n is an ideal on (X, τ) and for the ideal topological space (X, τ, I_n), $PIO(X) = PO(X, \tau^*(I))$ as $\tau \cap I_n = \{\emptyset\}$.

Theorem 2.4 (see [7]). *For a subset A of an ideal topological space (X, τ, I), A is pre-I-open if and if and only if $A = G \cap B$, where G is open and B is $*$-dense.*

If a set A is pre-I-closed (see [2]), then $X \setminus A$ is pre-I-open, i.e., $(X \setminus A) \subseteq Int(Cl^*(X \setminus A))$. Thus $(X \setminus A) \subseteq (X \setminus Cl(Int^*(A)))$ and hence $Cl(Int^*(A)) \subseteq A$.

It is obvious that every closed set is a pre-I-closed set. We denote by $PIC(X)$ the collection of all pre-I-closed sets in (X, τ, I). The family of all closed sets in (X, τ) is denoted by $C(\tau)$.

Theorem 2.5. *Finite intersection of pre-I-closed sets is a pre-I-closed set.*

Proof. Let (X, τ, I) be an ideal topological space. Let $P_1, P_2, \dots, P_n \in PIC(X)$. We shall prove that $\bigcap_{i=1}^n A_i \in PIC(X)$. For each i,

$$Cl(Int^*(\bigcap_{i=1}^{n} A_i)) = Cl(Int^*(A_1) \cap (Int^* A_2) \cap \cdots \cap (Int^*(A_n)) \subseteq Cl(Int^*(A_i)).$$

Thus

$$\begin{aligned} &Cl(Int^*(A_1) \cap (Int^* A_2) \cap \cdots \cap (Int^*(A_n)) \\ &\quad \subseteq Cl(Int^*(A_1)) \cap Cl(Int^* A_2)) \cap \cdots \cap Cl(Int^*(A_n)) \\ &\quad \subseteq A_1 \cap A_2 \cap \cdots \cap A_n. \end{aligned}$$

So $\bigcap_{i=1}^n A_i \in PIC(X)$. □

Theorem 2.6. *Let (X, τ, I) be an ideal topological space and let $A \in PIC(X)$. If $A \in \tau^*$, then $A \in C(\tau)$.*

Theorem 2.7. *Let (X, τ, I) be an ideal topological space. If $PIC(X) = \tau^*(I)$, then each open set in (X, τ^*) is closed in (X, τ).*

Proof. Let $A \in \tau^*$. Then $Int^*(A) = A$. Now $Cl(Int^*(A)) = Cl(A) \subseteq A$ as $A \in PIC(X)$. Thus A is closed in (X, τ^*). □

For the converse of the above theorem we get the following result.

Theorem 2.8. *Let (X, τ, I) be an ideal topological space. If each open set in (X, τ^*) is closed in (X, τ), then $\tau^* \subseteq PIC(X)$.*

Proof. Let $A \in \tau^*$. Since A is closed in (X, τ), $Cl(A) \subseteq A$ and hence $Cl(Int^*(A)) \subseteq A$. Thus $\tau^* \subseteq PIC(X)$. □

3. $\mathcal{C}_I$-continuity

Definition 3.1 (see [5])**.** A subset K of an ideal topological space (X, τ, I) is called a $\mathcal{C}_I$-set if $K = L \cap M$, where L is an open set and M is a pre-I-closed set in X.

Let $\mathcal{C}_I(X)$ denote the collection of all $\mathcal{C}_I$-sets in (X, τ, I).

Remark 3.2. (i) A subset S of (X, τ, I) is a $\mathcal{C}_I$-set if and only if $X \setminus S$ is the union of a closed set and a pre-I-open set.

(ii) Every open (respectively, pre-I-closed) subset of (X, τ, I) is a $\mathcal{C}_I$-set.

(iii) For any space (X, τ, I), $\mathcal{C}_I(X)$ is closed under finite intersections.

(iv) The complement of a $\mathcal{C}_I$-set need not be a $\mathcal{C}_I$-set. Hence the finite union of $\mathcal{C}_I$-sets need not be a $\mathcal{C}_I$-set.

Proof. (i) If $S = O \cap P$, where $O \in \tau,\ P \in PIC(X)$, then
$$X \setminus S = X \setminus (O \cap P) = (X \setminus O) \cup (X \setminus P).$$
Thus $(X \setminus O) \in C(\tau)$ and $(X \setminus P) \in PIO(X)$.

Conversely, suppose that $(X \setminus S) = F \cup P$, where $F \in C(\tau)$ and $P \in PIC(X)$. Now $S = (X \setminus F) \cap (X \setminus P)$. Since $(X \setminus F) \in \tau$ and $(X \setminus P) \in PIC(X)$, S is a $\mathcal{C}_I$-set.

(ii) Proof is obvious.

(iii) Proof is obvious from Theorem 2.5.

(iv) Consider the subset $S = \{1/n : n \in N\}$ of $(\mathbb{R}, \tau_{\mathbb{R}}, \{\emptyset\})$, where $\tau_{\mathbb{R}}$ is the usual topology on $\mathbb{R}$. Then S is a $\mathcal{C}_I$-set whereas $\mathbb{R} \setminus S$ is not. □

Definition 3.3 (see [1])**.** A subset S of a topological space (X, τ) is called locally closed if $S = U \cap F$, where $U \in \tau$ and $F \in C(\tau)$.

Remark 3.4. Since every closed set is a pre-I-closed set, every locally closed set is a $\mathcal{C}_I$-set. The converse of this implication is not true in general which will be shown in Example 3.10.

Definition 3.5 (see [5])**.** A subset K of an ideal topological space (X, τ, I) is an $\mathcal{A}_I^*$-set if $K = L \cap M$, where L is an open set and $M = Cl(Int^*(M))$.

Remark 3.6 (see [4])**.** Let (X, τ, I) be an ideal topological space. Any $\mathcal{A}_I^*$-set is locally closed in X. The reverse implication is not true in general.

Remark 3.7. Let (X, τ, I) be an ideal topological space. Any $\mathcal{A}_I^*$-set is a $\mathcal{C}_I$-set.

Proof. The proof follows from Remark 3.6. □

Definition 3.8 (see [5, 10])**.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is said to be

(1) $\mathcal{C}_I$-continuous if $f^{-1}(A)$ is a $\mathcal{C}_I$-set in X for every open set A in Y,

(2) LC-irresolute if $f^{-1}(M)$ is a locally closed set in X for each locally closed set M in Y,

(3) LC-continuous if $f^{-1}(V)$ is a locally closed set in X for each open set V in Y,

(4) sub-LC-continuous if there is a subbase (or, equivalently, a base) $\mathbb{B}$ for (Y, σ) such that $f^{-1}(V)$ is a locally closed set in X for each $V \in \mathbb{B}$.

From Remark 3.4, we have the following corollary.

Corollary 3.9. *Let $f : (X, \tau, I) \to (Y, \sigma)$ be a function. If f is LC-continuous, then it is $\mathcal{C}_I$-continuous.*

From the previous definition and corollary it follows immediately that we have the following implications:
$$\textit{continuous} \Longrightarrow \textit{LC-irresolute} \Longrightarrow \textit{LC-continuous}$$
$$\Longrightarrow \textit{sub-LC-continuous} \Longrightarrow \mathcal{C}_I\textit{-continuous}.$$

However, none of these implications can be reversed.

We shall give an example for the last implication only, other examples have been given by Ganster and Reilly [10].

Example 3.10. Let $X=\{a,b,c,d\}$, $\tau=\{X,\emptyset,\{a\},\{a,b\},\{c,d\},\{a,c,d\}\}$, and $I=\{\emptyset,\{a\},\{d\},\{a,d\}\}$. The function $f:(X,\tau,I)\to(X,\tau)$, defined by $f(a)=a,\ f(b)=a,\ f(c)=b,\ f(d)=a$, is $\mathcal{C}_I$-continuous but it is not sub-LC-continuous as $g^{-1}(a)=\{a,b,d\}$ is not a locally closed set in X.

From Proposition 10 of [10], we have that the composition of a sub-LC-continuous function and a continuous function need not be sub-LC-continuous. However, we have the following remark.

Remark 3.11. The composition of two $\mathcal{C}_I$-continuous functions need be a $\mathcal{C}_I$-continuous function in general.

Remark 3.12. It is also easy to verify that the composition of a continuous function and a $\mathcal{C}_I$-continuous function is $\mathcal{C}_I$-continuous.

Remark 3.13. The composition of a LC-continuous function and a $\mathcal{C}_I$-continuous function is $\mathcal{C}_I$-continuous.

A converse part of Corollary 3.9 follows from Theorem 2.6.

Theorem 3.14. *If each pre-I-closed in (X,τ,I) is open in $*$-topology, then avery $\mathcal{C}_I$-continuous function $f:(X,\tau,I)\to(Y,\sigma)$ is LC-continuous.*

To every function $f:X\to Y$ one can assign the graph function $g_f:X\to X\times Y$ defined by $g_f(x)=(x,f(x))$.

Theorem 3.15. *Let $f:(X,\tau,I)\to(Y,\sigma)$ be a function. If f is $\mathcal{C}_I$-continuous, then g_f is $\mathcal{C}_I$-continuous.*

Proof. Let $\mathbb{S}$ be a subbase for (Y,σ) such that $f^{-1}(V)\in\mathcal{C}_I(X)$ whenever $V\in\mathbb{S}$. Then $\{U\times V : U\in\tau,\ V\in\mathbb{S}\}$ is a subbase for the product topology on $X\times Y$. Since $g_f^{-1}(U\times V)=U\cap f^{-1}(V)$, g_f^{-1} is $\mathcal{C}_I$-continuous. □

4. Pre_I^*-continuity

Definition 4.1 (see [3])**.** A subset P of an ideal topological space (X,τ,I) is pre_I^*-open if $P\subseteq Int^*(Cl(P))$.

The collection of all pre_I^*-open sets is denoted by $P_I^*O(X)$.

Theorem 4.2. *Let (X,τ,I) be an ideal topological space, where I is co-dense. Then $PO(X,\tau)=P_I^*O(X)$.*

Proof. Let $P\in P_I^*O(X)$. Then

$$P\subseteq Int^*(Cl(P))\subseteq Cl(P)\cap\psi(Cl(P)).$$

Thus (see [11])

$$P \subseteq \psi(Cl(P)) = X \setminus (X \setminus Cl(P))^* = X \setminus Cl(X \setminus Cl(P)).$$

Now, since by Lemma 1.1

$$X \setminus Cl(X \setminus Cl(P)) = X \setminus (X \setminus (Int(Cl(P))),$$

we have that $P \in PO(X, \tau)$.

Reciprocally, suppose that $A \in PO(X, \tau)$. Then $A \subseteq Int(Cl(A)) \subseteq Int^*(Cl(A))$. Thus $A \in P_I^*O(X)$. □

If $I = \{\emptyset\}$ for an ideal topological space (X, τ, I), then $\tau \cap I = \{\emptyset\}$ and hence $P_I^*O(X) = PO(X, \tau)$.

For the ideal topological space (X, τ, I_n), $P_I^*O(X) = PO(X, \tau)$ as $\tau \cap I_n = \{\emptyset\}$.

Corollary 4.3. *Let (X, τ, I) be an ideal topological space with $\tau \cap I = \{\emptyset\}$. Then $A \in P_I^*O(X)$ if and only if it is of the form $D \cap O$, where D is dense and O is open in X.*

The complement of a pre_I^*-open set is called a pre_I^*-closed set (see [8, 3]). That is, if A is pre_I^*-closed in an ideal topological space (X, τ, I), then $X \setminus A \subseteq Int^*(Cl(X \setminus A))$. Thus $Cl^*(Int(A)) \subseteq A$ (see [8, 3]). The collection of all pre_I^*-closed sets is denoted by $P_I^*C(X)$.

Again, if A is closed in X, then it is obvious that A is a pre_I^*-closed set, since

$$Cl^*(Int(A)) \subseteq Cl(Int(A)) \subseteq Cl(A) = A.$$

Lemma 4.4. *Let (X, τ, I) be an ideal topological space. For any subset A of X, the following diagram holds:*

$$\begin{array}{ccccc} \textit{open} & \Longrightarrow & \textit{pre-}I\textit{-open} & \Longrightarrow & \textit{preopen in } (X,\tau) \\ & & \Downarrow & & \Downarrow \\ & & \textit{preopen in } (X,\tau^*) & \Longrightarrow & pre_I^*\textit{-open}. \end{array}$$

Corollary 4.5. *Let (X, τ, I) be an ideal topological space. For any subset A of X, the following diagram holds:*

$$\begin{array}{ccccc} \textit{closed} & \Longrightarrow & \textit{pre-}I\textit{-closed} & \Longrightarrow & \textit{preclosed in } (X,\tau) \\ & & \Downarrow & & \Downarrow \\ & & \textit{preclosed in } (X,\tau^*) & \Longrightarrow & pre_I^*\textit{-closed}. \end{array}$$

Interesting results have been drawn from Proposition 2.3 and Theorem 4.2.

Remark 4.6. If I is codense then the following diagram holds:

$$\begin{array}{ccccc} \textit{open} & \Longrightarrow & \textit{pre-}I\textit{-open} & \Longrightarrow & \textit{preopen in } (X,\tau) \\ & & \Updownarrow & & \Updownarrow \\ & & \textit{preopen in } (X,\tau^*) & \Longrightarrow & pre^*_I\textit{-open}. \end{array}$$

Therefore, preopenness in (X, τ^*) implies preopenness in (X, τ) but the converse is not true even if I is codense: see the next example.

Example 4.7. Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$, and $I = \{\emptyset, \{b\}\}$. Then I is codense. Here $\{b\}$ is preopen in (X, τ) but it is not preopen in (X, τ^*).

Next example shows that the condition "I is codense" is an essential condition for the statement: preopenness in (X, τ^*) implies preopenness in (X, τ).

Example 4.8. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, and $I = \{\emptyset, \{a\}\}$. Then I is not a codense ideal and $\tau^* = \{\emptyset, X, \{a\}, \{b, c\}\}$. If $A = \{b, c\}$, then $Int(Cl(\{b, c\})) = \emptyset$ but $Int^*(Cl^*(\{b, c\})) = \{b, c\}$. Thus $\{b, c\}$ is preopen in (X, τ^*) but not preopen in (X, τ).

Definition 4.9 (see [6, 2, 14])**.** A function $f : (X, \tau, I) \to (Y, \sigma)$ is called pre^*_I-continuous (respectively, pre-I-continuous, pre-continuous) if $f^{-1}(T)$ is a pre^*_I-closed (respectively, pre-I-open, preopen) subset of X for each closed (respectively, open, open) subset T of Y.

Theorem 4.10. *If a function $f : (X, \tau, I) \to (Y, \sigma)$ is continuous, then it is a pre^*_I-continuous function.*

Proof. Proof is obvious from Lemma 4.4. □

The composition of two pre^*_I-continuous functions is not a pre^*_I-continuous function in general.

Remark 4.11. The composition of a continuous function and a pre^*_I-continuous function is pre^*_I-continuous.

Theorem 4.12. *A function $f : (X, \tau, I) \to (Y, \sigma)$ is pre-I-continuous if and only if $f^{-1}(F)$ is pre-I-closed for each closed set F in Y.*

Proof. Proof is obvious from Corollary 4.5. □

Corollary 4.13. *Every pre-I-continuous function $f : (X, \tau, I) \to (Y, \sigma)$ is a pre^*_I-continuous function.*

Acknowledgement. The authors are thankful to the referees for valuable comments.

References

[1] N. Bourbaki, *Elements of Mathematics.General Topology. Part I*, Hermann, Paris; Addlson-Wesley, Reading, Mass. – London – Don Mills, 1966.

[2] J. Dontchev, *Idealization of Ganster-Reilly decomposition theorems*, arxiv: math. GN/9901017v1 (1999).

[3] E. Ekici, *On $\mathcal{AC}_I$-sets, $\mathcal{BC}_I$-sets, β_I^*-open sets and decompositions of continuity in ideal topological spaces*, Creat. Math. Inform. **20** (2011), 47–54.

[4] E. Ekici, *On R-I-open sets and $\mathcal{A}_I^*$-sets in ideal topological spaces*, An. Univ. Craiova Ser. Mat. Inform. **38** (2) (2011), 26–31.

[5] E. Ekici, *On $\mathcal{A}_I^*$-sets, $\mathcal{C}_I$-sets, $\mathcal{C}_I^*$-sets and decompositions of continuity in ideal topological spaces*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N. S.) **59** (2013), 173–184.

[6] E. Ekici and O. Elmali, *On decompositions via generalized closedness in ideal spaces*, Filomat **29**(4) (2015), 879–886.

[7] E. Ekici and T. Noiri, *Certain subsets in ideal topological spaces*, An. Univ. Oradea Fasc. Mat. **17** (2011), 125–132.

[8] E. Ekici and S. Özen, *A generalized class of τ^* in ideal spaces*, Filomat **27**(4) (2013), 527–535.

[9] M. Ganster, *Preopen sets and resolvable spaces*, Kyungpook Math. J. **27**(2) (1987), 135–143.

[10] M. Ganster and I. L. Reilly, *Locally closed sets and LC-continuous functions*, Int. J. Math. Math. Sci. **12**(3) (1989), 417–424.

[11] T. R. Hamlett and D. Janković, *Ideals in topological spaces and the set operator ψ*, Boll. Un. Mat. Ital. B(7) **4** (1990), 863–874.

[12] D. Janković and T. R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly **97** (1990), 295–310.

[13] K. Kuratowski, *Topology. Vol. I.* Academic Press, New York, 1966.

[14] A. S. Mashhour, M. E. Abd. El-Monsef, and S. N. El-Deeb, *On precontinous and week precontinous mappings*, Proc. Math. Phys. Soc. Egypt **53** (1982), 47–53.

[15] S. Modak, *A study on structure and functions in topological spaces with ideals*, Thesis, Burd. Univ. India, 2008.

[16] S. Modak and C. Bandyopadhyay, *$*$-topology and generalized open sets*, Soochow J. Math. **32** (2006), 201–210.

[17] W. J. Thron, *Topological Structures*, Holt, Rinehart and Winston, New York, 1966.

Department of Mathematics, University of Gour Banga, P.O. Mokdumpur, Malda - 732103, India

E-mail address: spmodak2000@yahoo.co.in

2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumomoto-ken, 869-5142 Japan

E-mail address: t.noiri@nifty.com