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Stability in higher-order nonlinear fractional differential equations

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Abstract. We give sufficient conditions to guarantee the asymptotic stability of the zero solution to a kind of higher-order nonlinear fractional differential equations. By using Krasnoselskii's fixed point theorem in a weighted Banach space, we establish new results on the asymptotic stability of the zero solution provided that $f(t,0) = 0$. The results obtained here generalize the work of Ge and Kou [6].

1. Introduction

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[16], [18], and the references therein.

In this paper, we are interested in the analysis of the qualitative theory of problems of the asymptotic stability of the zero solution to higher-order fractional differential equations. Inspired and motivated by the works mentioned above, we concentrate on the asymptotic stability of the zero solution

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for the higher-order nonlinear fractional differential equation

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x(t) = f\left(t, x(t)\right), \ \ t \geq 0, \\ x^{(i)}(0) = x_i, \ \ i = 0, \dots, n, \end{cases} \tag{1.1}$$

where $n < \alpha < n+1$, $n \geq 1$, $\mathbb{R}^+ = [0, +\infty)$, $x_i \in \mathbb{R}$, $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that the fractional integral $I_{0+}^{\alpha-1} f\left(t, x(t)\right)$ exists and $f(t, 0) = 0$, and ${}^{C}D_{0+}^{\alpha}$ is the standard Caputo fractional derivative. We denote the solution of (1.1) by $x\left(t, x_0, \dots, x_n\right)$. To show the asymptotic stability of the zero solution, we transform (1.1) into an integral equation and then use Krasnoselskii's fixed point theorem. The obtained integral equation is the sum of two mappings, one is a contraction and the other is compact. In the case $n = 1$, Ge and Kou [6] show the asymptotic stability of the zero solution of (1.1) by employing Krasnoselskii's fixed point theorem in a weighted Banach space.

This paper is organized as follows. In Section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later sections. Also, we present the inversion of (1.1) and Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [17]. In Section 3, we give and prove our main results on stability. The results obtained here generalize the work of Ge and Kou [6].

2. Preliminaries

We introduce some necessary definitions, lemmas, and theorems which will be used in this paper. For more details, see [7, 9, 16, 17].

Definition 2.1 (see [7, 16])**.** The fractional integral of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Definition 2.2 (see [7, 16])**.** The Caputo fractional derivative of order $\alpha > 0$ of a function $x : \mathbb{R}^+ \to \mathbb{R}$ is given by

$${}^{C}D_{0+}^{\alpha} x(t) = I_{0+}^{n-\alpha} x^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.3 (see [7, 16])**.** *Let* $\Re(\alpha) > 0$. *Suppose that* $x \in C^{n-1}[0, +\infty)$ *and* $x^{(n)}$ *exists almost everywhere on any bounded interval of* $\mathbb{R}^+$. *Then*

$$\left(I_{0+}^{\alpha}\, {}^{C}D_{0+}^{\alpha} x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} t^k.$$

In particular, when $0 < \Re(\alpha) < 1$, *we get*

$$\left(I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}x\right)(t) = x(t) - x(0).$$

Remark 2.4. From Definitions 2.1, 2.2, and Lemma 2.3, it is easy to see that

(1) Let $\Re(\alpha) > 0$. If x is continuous on $\mathbb{R}^+$, then $D_{0+}^{\alpha}I_{0+}^{\alpha}x(t) = x(t)$ holds for all $t \in \mathbb{R}^+$;

(2) The Caputo derivative of a constant is equal to zero.

The following Banach space plays a fundamental role in our discussion. Let $h : [0,+\infty) \to [1,+\infty)$ be a strictly increasing continuous function with $h(0) = 1$, $h(t) \to \infty$ as $t \to \infty$, and $h(s)h(t-s) \le h(t)$ for all $0 \le s \le t \le \infty$. Let

$$E = \left\{x \in C\left([0,+\infty)\right) : \sup_{t\ge 0} |x(t)|/h(t) < \infty\right\}.$$

Then E is a Banach space equipped with the norm

$$\|x\| = \sup_{t\ge 0} \frac{|x(t)|}{h(t)}.$$

For more properties of this Banach space, see [9]. Moreover, let

$$\|\varphi\|_t = \max\left\{|\varphi(s)| : 0 \le s \le t\right\},$$

for any $t \ge 0$, any given $\varphi \in C\left([0,+\infty)\right)$, and let $\Im(\varepsilon) = \{x \in E : \|x\| \le \varepsilon\}$ for any $\varepsilon > 0$.

Lemma 2.5 (see [6])**.** *Let* $r \in C\left([0,+\infty)\right)$*. Then* $x \in C\left([0,+\infty)\right)$ *is a solution of the Cauchy type problem*

$$\begin{cases} {}^{C}D_{0+}^{\alpha}x(t) = r(t),\ t \in \mathbb{R}^+,\ \ n < \alpha < n+1, \\ x^{(i)}(0) = x_i,\ \ i = 0,\ldots,n, \end{cases}$$

if and only if x *is a solution of the Cauchy type problem*

$$\begin{cases} x'(t) = I_{0+}^{\alpha-1}r(t) + \sum\limits_{i=1}^{n} \frac{x_i}{(i-1)!}t^{i-1},\ \ t \in \mathbb{R}^+, \\ x(0) = x_0. \end{cases}$$

Lemma 2.6. *Let* $k \in \mathbb{R}$*. Then* $x \in C\left([0,+\infty)\right)$ *is a solution of* (1.1) *if and only if*

$$\begin{aligned} x(t) &= e^{-kt}x_0 + \sum_{i=1}^{n} x_i g_i(t) + k\int_0^t e^{-k(t-u)}x(u)du \\ &\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t\int_u^t e^{-k(t-s)}(s-u)^{\alpha-2}ds f(u,x(u))du, \end{aligned} \tag{2.1}$$

where

$$g_i\left(t\right)=\sum_{j=1}^{i}\frac{\left(-1\right)^{j-1}t^{i-j}}{(i-j)!k^j}+\frac{\left(-1\right)^i}{k^i}e^{-kt},\ \ i=1,\ldots,n.$$

Proof. Let $x\in C\left([0,+\infty)\right)$ be a solution of (1.1). From Lemma 2.5, we have

$$\begin{cases} x'(t)=I_{0+}^{\alpha-1}\left(f\left(t,x\left(t\right)\right)\right)+\sum\limits_{i=1}^{n}\frac{x_i}{(i-1)!}t^{i-1},\ t\in\mathbb{R}^+,\\ x(0)=x_0.\end{cases}$$

Then

$$\begin{cases} x'(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f(s,x(s))ds+\sum\limits_{i=1}^{n}\frac{x_i}{(i-1)!}t^{i-1},\ t\in\mathbb{R}^+,\\ x(0)=x_0.\end{cases}$$

Rewriting this as

$$\begin{cases} x'(t)+kx\left(t\right)\\ =kx\left(t\right)+\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f(s,x(s))ds+\sum\limits_{i=1}^{n}\frac{x_i}{(i-1)!}t^{i-1},\ t\in\mathbb{R}^+,\\ x(0)=x_0,\end{cases}$$

by the variation of constants formula we obtain (2.1). Since each step is reversible, the converse follows easily. This completes the proof. □

Definition 2.7. The trivial solution $x=0$ of (1.1) is said to be *stable* in the Banach space E, if for every $\varepsilon>0$, there exists a $\delta=\delta\left(\varepsilon\right)>0$ such that $\sum_{i=0}^{n}\left|x_i\right|\leq\delta$ implies that the solution $x(t)=x(t,x_0,\ldots,x_n)$ exists for all $t\geq0$ and satisfies $\|x\|\leq\varepsilon$, and *asymptotically stable*, if it is stable in E and there exists a number $\sigma>0$ such that $\sum_{i=0}^{n}\left|x_i\right|\leq\sigma$ implies $\lim_{t\to\infty}\|x(t)\|=0$.

We end this section by stating Krasnoselskii's fixed point theorem which enables us to prove the asymptotic stability of the zero solution to (1.1). For its proof we refer the reader to [17].

Theorem 2.8 (Krasnoselskii, see [17])**.** *Let Ω be a non-empty closed convex subset of a Banach space $(S,\|.\|)$. Suppose that A and B map Ω into S so that*

(i) *$Ax+By\in\Omega$ for all $x,y\in\Omega$,*
(ii) *A is continuous and $A\Omega$ is contained in a compact set of S,*
(iii) *B is a contraction with constant $l<1$.*

Then there is an $x\in\Omega$ with $Ax+Bx=x$.

In order to prove (ii), the following modified compactness criterion is needed.

Theorem 2.9 (see [9]). *Let $\mathcal{M}$ be a subset of the Banach space E. Then $\mathcal{M}$ is relatively compact in E if the following conditions are satisfied:*

(i) $\{x(t)/h(t) : x \in \mathcal{M}\}$ *is uniformly bounded,*

(ii) $\{x(t)/h(t) : x \in \mathcal{M}\}$ *is equicontinuous on any compact interval of* $\mathbb{R}^+$,

(iii) $\{x(t)/h(t) : x \in \mathcal{M}\}$ *is equiconvergent at infinity, i.e., for any given $\varepsilon > 0$, there exists a $T_0 > 0$ such that for all $x \in \mathcal{M}$ and $t_1, t_2 > T_0$,*

$$|x(t_2)/h(t_2) - x(t_1)/h(t_1)| < \varepsilon.$$

3. Main results

Before stating and proving the main results, we introduce the following three hypotheses.

(H1) f is a continuous function and $f(t, 0) = 0$.

(H2) There exists a constant $\beta_1 \in (0, 1)$ such that

$$\begin{aligned} &e^{-kt}/h(t) \in BC\left([0, +\infty)\right) \cap L^1\left([0, +\infty)\right), \\ &|k| \int_0^\infty e^{-ku}/h(u)du \le \beta_1 < 1. \end{aligned} \tag{3.1}$$

(H3) There exist constants $\eta > 0$, $\beta_2 \in (0, 1 - \beta_1)$, and a continuous function $\tilde{f} : [0, \infty) \times (0, \eta] \to \mathbb{R}^+$ such that

$$\frac{|f(t, \upsilon h(t))|}{h(t)} \le \tilde{f}(t, |\upsilon|) \tag{3.2}$$

holds for all $t \ge 0$, $0 < |\upsilon| \le \eta$, and

$$\sup_{t \ge 0} \int_0^t \frac{K(t-u)}{h(t-u)} \frac{\tilde{f}(u, r)}{r} du \le \beta_2 < 1 - \beta_1 \tag{3.3}$$

holds for every $0 < r \le \eta$, where $\tilde{f}(t, r)$ is nondecreasing in r for fixed t, $\tilde{f}(t, r) \in L^1\left([0, +\infty)\right)$ in t for fixed r, and

$$K(t-u) = \begin{cases} \frac{1}{\Gamma(\alpha-1)} \int_u^t e^{-k(t-s)}(s-u)^{\alpha-2} ds, & t-u \ge 0, \\ 0, & t-u < 0. \end{cases}$$

Theorem 3.1. *Suppose that* (H1)$-$(H3) *hold. Then the trivial solution $x = 0$ of* (1.1) *is stable in the Banach space E.*

Proof. For any given $\varepsilon > 0$, we first prove the existence of a $\delta > 0$ such that

$$\textstyle\sum_{i=0}^n |x_i| < \delta \text{ implies } \|x\| \le \varepsilon.$$

In fact, according to (3.1), there exist constants M_1 and $M > 0$ such that

$$\frac{e^{-kt}}{h(t)} \le M_1 \text{ and } \frac{|g_i(t)|}{h(t)} \le M, \ \ i = 1, \ldots, n.$$

Let

$$0 < \delta \leq \frac{1-\beta_1-\beta_2}{M_1+M}\varepsilon.$$

Consider the non-empty closed convex subset $\Im(\varepsilon) \subseteq E$. For $t \geq 0$, we define two mapping A and B on $\Im(\varepsilon)$ as follows:

$$\begin{aligned} Ax(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^t \int_u^t e^{-k(t-s)}(s-u)^{\alpha-2} ds f(u, x(u)) du \\ &= \int_0^t K(t-u) f(u, x(u)) du \end{aligned}$$

and

$$Bx(t) = e^{-kt}x_0 + \sum_{i=1}^{n} x_i g_i(t) + k\int_0^t e^{-k(t-u)} x(u) du.$$

Obviously, for $x \in \Im(\varepsilon)$, both Ax and Bx are continuous functions on $[0, +\infty)$. Furthermore, for $x \in \Im(\varepsilon)$, by (3.1)–(3.3), for any $t \geq 0$, we have

$$\begin{aligned} \frac{|Ax(t)|}{h(t)} &\leq \int_0^t \frac{K(t-u)}{h(t-u)} \frac{|f(u,x(u))|}{h(u)} du \\ &\leq \int_0^t \frac{K(t-u)}{h(t-u)} \tilde{f}\left(u, \frac{|x(u)|}{h(u)}\right) du \\ &\leq \beta_2 \|x\| \leq \beta_2 \varepsilon < \infty \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} \frac{|Bx(t)|}{h(t)} &= \left| \frac{e^{-kt}}{h(t)} x_0 + \sum_{i=1}^{n} x_i \frac{g_i(t)}{h(t)} + k\int_0^t \frac{e^{-k(t-u)}}{h(t)} x(u) du \right| \\ &\leq M_1 |x_0| + M \sum_{i=1}^{n} |x_i| + |k| \int_0^\infty \frac{e^{-ku}}{h(u)} du \, \|x\| \qquad (3.5) \\ &\leq M_1 |x_0| + M \sum_{i=1}^{n} |x_i| + \beta_1 \varepsilon < \infty. \end{aligned}$$

Then $A\Im(\varepsilon) \subseteq E$ and $B\Im(\varepsilon) \subseteq E$. Next, we shall use Theorem 2.8 to prove that there exists at least one fixed point of the operator $A+B$ in $\Im(\varepsilon)$. Here, we divide the proof into three steps.

Step 1. We prove that $Ax + By \in \Im(\varepsilon)$ for all $x, y \in \Im(\varepsilon)$. Indeed, for any $x, y \in \Im(\varepsilon)$, from (3.4) and (3.5) we obtain that

$$\begin{aligned} &\sup_{t\geq 0} \frac{|Ax(t) + By(t)|}{h(t)} \\ &= \sup_{t\geq 0} \left| \frac{e^{-kt}}{h(t)} x_0 + \sum_{i=1}^{n} x_i \frac{g_i(t)}{h(t)} + k\int_0^t \frac{e^{-k(t-u)}}{h(t)} y(u) du \right. \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \frac{K(t-u)}{h(t)} f(u,x(u))du\Bigg| \\
&\le M_1\,|x_0| + M\sum_{i=1}^{n}|x_i| + |k|\int_0^\infty \frac{e^{-ku}}{h(u)}du\,\|y\| + \beta_2\,\|x\| \\
&\le (M_1+M)\,\delta + \beta_1\varepsilon + \beta_2\varepsilon \le \varepsilon,
\end{aligned}
$$

which implies that $Ax+By \in \Im(\varepsilon)$ for all $x, y \in \Im(\varepsilon)$.

Step 2. It is easy to see that A is continuous. Now we only prove that $A\Im(\varepsilon)$ is relatively compact in E. In fact, from (3.4), we get that $\{x(t)/h(t) : x \in \Im(\varepsilon)\}$ is uniformly bounded in E. Moreover, a classical theorem states the fact that the convolution of an L^1-function with a function tending to zero, does also tend to zero. Then we conclude that, for $t-u \ge 0$, we have

$$
\begin{aligned}
0 \le \lim_{t\to\infty}\frac{K(t-u)}{h(t-u)} &\le \lim_{t\to\infty}\frac{1}{\Gamma(\alpha-1)}\int_u^t \frac{e^{-k(t-s)}}{h(t-u)}\frac{(s-u)^{\alpha-2}}{h(s-u)}ds \\
&= \lim_{t\to\infty}\frac{1}{\Gamma(\alpha-1)}\int_0^t \frac{e^{-k(t-u-s)}}{h(t-u-s)}\frac{s^{\alpha-2}}{h(s)}ds = 0,
\end{aligned}
$$

due to the fact $\lim_{t\to\infty} t^{\alpha-2}/h(t) = 0$. Together with the continuity of K and h, we get that there exists a constant $M_2 > 0$ such that

$$
\left|\frac{K(t-u)}{h(t-u)}\right| \le M_2, \tag{3.6}
$$

and for any $T_0 \in \mathbb{R}^+$, the function $K(t-u)h(u)/h(t)$ is uniformly continuous on $\{(t,u) : 0 \le u \le t \le T_0\}$. Now, for any $t_1, t_2 \in [0, T_0]$, $t_1 < t_2$, we have

$$
\begin{aligned}
&\left|\frac{Ax(t_2)}{h(t_2)} - \frac{Ax(t_1)}{h(t_1)}\right| \\
&= \left|\int_0^{t_2}\frac{K(t_2-u)}{h(t_2)}f(u,x(u))du - \int_0^{t1}\frac{K(t_1-u)}{h(t_1)}f(u,x(u))du\right| \\
&\le \int_0^{t_1}\left|\frac{K(t_2-u)}{h(t_2)} - \frac{K(t_1-u)}{h(t_1)}\right| |f(u,x(u))|\,du \\
&\quad + \int_{t_1}^{t_2}\frac{K(t_2-u)}{h(t_2-u)}\tilde{f}(u,\varepsilon)\,du \\
&\le \int_0^{t_1}\left|\frac{K(t_2-u)h(u)}{h(t_2)} - \frac{K(t_1-u)h(u)}{h(t_1)}\right| \tilde{f}(u,\varepsilon)\,du \\
&\quad + M_2\int_{t_1}^{t_2}\tilde{f}(u,\varepsilon)\,du \;\to\; 0,
\end{aligned}
$$

as $t_2 \to t_1$, which means that $\{x(t)/h(t) : x \in \Im(\varepsilon)\}$ is equicontinuous on any compact interval of $\mathbb{R}^+$. By Theorem 2.9, in order to show that $A\Im(\varepsilon)$ is a relatively compact set of E, we only need to prove that $\{x(t)/h(t) : x \in \Im(\varepsilon)\}$ is equiconvergent at infinity. In fact, for any $\varepsilon_1 > 0$, there exists a $L > 0$ such that

$$M_2 \int_L^\infty \tilde{f}(u,\varepsilon)\, du \leq \frac{\varepsilon_1}{3}.$$

According to (3.6), we get that

$$\lim_{t\to\infty} \sup_{u\in[0,L]} \frac{K(t-u)}{h(t-u)} \leq \max\left\{\lim_{t\to\infty} \frac{K(t-L)}{h(t-L)}, \lim_{t\to\infty} \frac{K(t)}{h(t)}\right\} = 0.$$

Thus, there exists $T > L$ such that for $t_1, t_2 \geq T$ we have

$$\begin{aligned}
&\sup_{u\in[0,L]} \left| \frac{K(t_2-u)h(u)}{h(t_2)} - \frac{K(t_1-u)h(u)}{h(t_1)} \right| \\
&\leq \sup_{u\in[0,L]} \left| \frac{K(t_2-u)}{h(t_2-u)} \right| + \sup_{u\in[0,L]} \left| \frac{K(t_1-u)}{h(t1-u)} \right| \\
&\leq \frac{\varepsilon_1}{3} \left(\int_0^\infty \tilde{f}(u,\varepsilon)\, du \right)^{-1}
\end{aligned}$$

Therefore, if $t_1, t_2 \geq T$, then

$$\begin{aligned}
&\left| \frac{Ax(t_2)}{h(t_2)} - \frac{Ax(t_1)}{h(t_1)} \right| \\
&= \left| \int_0^{t_2} \frac{K(t_2-u)}{h(t_2)} f(u,x(u))du - \int_0^{t1} \frac{K(t_1-u)}{h(t_1)} f(u,x(u))du \right| \\
&\leq \int_0^L \left| \frac{K(t_2-u)h(u)}{h(t_2)} - \frac{K(t_1-u)h(u)}{h(t_1)} \right| \tilde{f}(u,\varepsilon)\, du \\
&\quad + \int_L^{t_2} \frac{K(t_2-u)}{h(t_2-u)} \tilde{f}(u,\varepsilon)\, du + \int_L^{t_1} \frac{K(t_1-u)}{h(t_1-u)} \tilde{f}(u,\varepsilon)\, du \\
&\leq \frac{\varepsilon_1}{3} + 2M_2 \int_L^\infty \tilde{f}(u,\varepsilon)\, du \leq \varepsilon_1.
\end{aligned}$$

Hence the required conclusion is true.

Step 3. We claim that $B : \Im(\varepsilon) \to E$ is a contraction mapping. In fact, for any $x, y \in \Im(\varepsilon)$, from (3.1) we obtain that

$$\begin{aligned}
&\sup_{t\geq 0} \left| \frac{Bx(t)}{h(t)} - \frac{By(t)}{h(t)} \right| \\
&= \sup_{t\geq 0} \left| k \int_0^t \frac{e^{-k(t-u)}}{h(t)} x(u)du - k \int_0^t \frac{e^{-k(t-u)}}{h(t)} y(u)du \right| \\
&\leq \sup_{t\geq 0} \left\{ |k| \int_0^t \frac{e^{-k(t-u)}}{h(t-u)} \frac{|x(u)-y(u)|}{h(u)} du \right\}
\end{aligned}$$

$$\leq |k| \int_0^t \frac{e^{-k(t-u)}}{h(t-u)} du \, \|x - y\|$$
$$\leq \beta_1 \|x - y\| .$$

By Theorem 2.8, we know that there exists at least one fixed point of the operator $A + B$ in $\Im(\varepsilon)$. Finally, for any $\varepsilon_2 > 0$, if

$$0 < \delta_1 \leq \frac{1 - \beta_1 - \beta_2}{M_1 + M} \varepsilon_2,$$

then $\sum_{i=0}^{n} |x_i| \leq \delta_1$ implies that

$$\|x\| = \sup_{t \geq 0} \left| \frac{e^{-kt}}{h(t)} x_0 + \sum_{i=1}^{n} x_i \frac{g_i(t)}{h(t)} + k \int_0^t \frac{e^{-k(t-u)}}{h(t)} x(u) du \right.$$
$$\left. + \int_0^t \frac{K(t-u)}{h(t)} f(u, x(u)) du \right|$$
$$\leq \sup_{t \geq 0} \left\{ \frac{e^{-kt}}{h(t)} |x_0| + \sum_{i=1}^{n} |x_i| \frac{|g_i(t)|}{h(t)} + |k| \int_0^\infty \frac{e^{-k(t-u)}}{h(t-u)h(u)} |x(u)| \, du \right.$$
$$\left. + \int_0^t \frac{K(t-u)}{h(t-u)} \frac{|f(u, x(u))|}{h(u)} du \right\}$$
$$\leq (M_1 + M) \delta_1 + \beta_1 \|x\| + \beta_2 \|x\|$$
$$\leq \frac{M_1 + M}{1 - \beta_1 - \beta_2} \delta_1 \leq \varepsilon_2.$$

Thus, we know that the trivial solution of (1.1) is stable in the Banach space E. □

Theorem 3.2. *Suppose that all conditions of Theorem 3.1 are satisfied,*

$$\lim_{t \to \infty} e^{-kt}/h(t) = 0, \tag{3.7}$$

and for any $r > 0$, there exists a function $\varphi_r(t) \in L^1([0, +\infty))$, $\varphi_r(t) > 0$, such that $|u| \leq r$ implies

$$|f(t, u)| / h(t) \leq \varphi_r(t) \text{ for a.e. } t \in [0, +\infty) .$$

Then the trivial solution of (1.1) *is asymptotically stable.*

Proof. First, it follows from Theorem 3.1 that the trivial solution of (1.1) is stable in the Banach space E. Next, we shall show that the trivial solution $x = 0$ of (1.1) is attractive. For any $r > 0$, define

$$\Im_*(r) = \left\{ x \in \Im(r) , \lim_{t \to \infty} x(t)/h(t) = 0 \right\} .$$

We only need to prove that $Ax + By \in \Im_*(r)$ for any $x, y \in \Im_*(r)$, i.e.,

$$\frac{Ax(t) + By(t)}{h(t)} \to 0 \text{ as } t \to \infty,$$

where

$$\begin{aligned} Ax(t)+By(t) &= e^{-kt}x_0+\sum_{i=1}^{n}x_i g_i\left(t\right)+k\int_0^t e^{-k(t-u)}y(u)du \\ &\quad +\int_0^t K(t-u)f(u,x(u))du. \end{aligned}$$

In fact, for $x, y \in \Im_*\left(r\right)$, based on the fact that was used in the proof of Theorem 3.1 (Step 2), it follows from (3.1) and (3.7) that

$$\int_0^t \frac{e^{-k(t-u)}}{h(t-u)}\frac{y(u)}{h(u)}du \to 0,$$

and

$$\frac{K(t-u)}{h(t-u)}=\frac{\int_u^t \frac{e^{-k(t-s)}}{h(t-u)}(s-u)^{\alpha-2}ds}{\Gamma(\alpha-1)} \to 0,$$

as $t \to \infty$. Together with the hypothesis $\varphi_r(t) \in L^1\left([0,+\infty)\right)$, we obtain that

$$\int_0^t \frac{K(t-u)}{h(t-u)}\frac{|f(u,x(u))|}{h(u)}du \leq \int_0^t \frac{K(t-u)}{h(t-u)}\varphi_r(u)du \to 0,$$

as $t \to \infty$. Thus we get the conclusion. □

By a similar argument as in Theorem 3.2, we obtain the following corollary. We omit the details.

Corollary 3.3. *Suppose that all conditions of Theorem 3.1 are satisfied and* (3.7) *holds for any* $r > 0$. *If there exist two functions* $a, b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *satisfying*

$$a(t) \in L^1(0,+\infty), \quad \lim_{r\to 0} b(r)/r = 0$$

such that

$$\left|f(t,u)\right|/h(t) \leq a(t)b(|u|),$$

then the trivial solution of (1.1) *is asymptotically stable.*

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