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# Complete asymptotics of the approximation of function from the Sobolev classes by the Poisson integrals

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Abstract. We obtain the complete asymptotic expansions of the least upper bounds of approximation of functions from the classes $W^r_\infty$ and $W^r_1$, $r \in \mathbb{N}$, by their Poisson integrals in the uniform and integral metrics.

## 1. Problem statement

Let $C$ be the space of $2\pi$-periodic continuous functions with the norm $\|f\|_C = \max_t |f(t)|$, let $L_\infty$ be the space of $2\pi$-periodic measurable essentially bounded functions with the norm $\|f\|_\infty = \operatorname{ess\,sup}_t |f(t)|$, and let $L$ be the space of $2\pi$-periodic Lebesgue summable on the period functions, in which the norm is $\|f\|_L = \|f\|_1 = \int_{-\pi}^{\pi} |f(t)| dt$.

We consider a boundary value problem in a unit disk for the equation

$$\Delta u = 0, \tag{1}$$

where $\Delta$ is the Laplace operator in the polar coordinates. Thus the equation (1) can be rewritten as follows:

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial x^2} = 0 \quad (0 \le \rho < 1,\ -\pi \le x \le \pi). \tag{2}$$

The solution of the equation (2) that satisfies the boundary condition

$$u\,(\rho, x)\,\Big|_{\rho=1} = f(x), \quad -\pi \le x \le \pi, \tag{3}$$

where $f$ is a Lebesgue summable $2\pi$–periodic function, will be denoted by $P(\rho; f; x) = u(\rho, x)$. Then the solution of the boundary problem (2)–(3)

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could be represented as follows:

$$P(\rho;f;x)=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(t+x)\Big\{\frac{1}{2}+\sum_{k=1}^{\infty}\rho^k\cos kt\Big\}dt,\quad 0\le\rho<1. \tag{4}$$

The equality (4) is referred as the Poisson integral of the function $f$. Assuming that $\rho=e^{-\frac{1}{\delta}}$ we will rewrite the Poisson integral by the means of the formula

$$P_\delta(f;x)=\frac{1}{\pi}\int\limits_{-\pi}^{\pi}f(t+x)\Big\{\frac{1}{2}+\sum_{k=1}^{\infty}e^{-\frac{k}{\delta}}\cos kt\Big\}dt,\quad \delta>0.$$

We denote by $W_p^r$, $p=1,\,\infty$, the class of $2\pi$-periodic functions $f$ having absolutely continuous derivatives including to $(r-1)$-th order and $\|f^{(r)}\,(t)\,\|_p\le 1$. Classes $W_p^r$ are called the Sobolev classes.

Let us consider the quantities

$$\mathcal{E}\,(\mathfrak{N};P_\delta)_C=\sup_{f\in\mathfrak{N}}\|f\,(\cdot)-P_\delta\,(f;\cdot)\|_C, \tag{5}$$

$$\mathcal{E}\,(\mathfrak{N};P_\delta)_1=\sup_{f\in\mathfrak{N}}\|f\,(\cdot)-P_\delta\,(f;\cdot)\|_1. \tag{6}$$

If the function $g\,(\delta)=g\,(\mathfrak{N};\delta)$ is found in an explicit form such that

$$\mathcal{E}\,(\mathfrak{N};P_\delta)_X=g\,(\delta)+o\,(g\,(\delta))\,,\quad \delta\to\infty,$$

then, in consistence with Stepanets [11], we will say that Kolmogorov–Nikolsky problem for given class $\mathfrak{N}$ and Poisson integral $P_\delta(f;x)$ is solved in the metric of space $X$.

A formal series $\sum_{n=0}^{\infty}g_n\,(\delta)$ will be called the complete asymptotic expansion or the complete asymptotics of the functions $f\,(\delta)$ as $\delta\to\infty$, if for all $n\in\mathbb{N}$,

$$|g_{n+1}\,(\delta)|=o\,(|g_n\,(\delta)|)\,, \tag{7}$$

and for every natural $m$,

$$f\,(\delta)=\sum_{n=0}^{m}g_n\,(\delta)+o\,(g_m\,(\delta))\,,\quad \delta\to\infty. \tag{8}$$

Briefly, this fact could be written as follows:

$$f\,(\delta)\cong\sum_{n=0}^{\infty}g_n\,(\delta).$$

The purpose of this work is to obtain the complete asymptotic expansions of the quantities (5) and (6) settings $\mathfrak{N}=W_p^r,\ p=1,\infty$, in the terms of $1/\delta$ as $\delta\to\infty$ for any $r\in\mathbb{N}$.

We denote by $K_n$ and $\tilde{K}_n$ the well-known constants of Favard–Akhiezer–Krein:

$$K_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{m(n+1)}}{(2m+1)^{n+1}}, \quad n = 0, 1, 2, \ldots,$$

$$\tilde{K}_n = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^{mn}}{(2m+1)^{n+1}}, \quad n \in \mathbb{N}.$$

## 2. Complete asymptotic expansions of the approximation by Poisson integrals on the classes $W^r_\infty$ and $W^r_1$

The first results that are connected with the research of the quantity $\mathcal{E}\left(W^r_\infty; P(\rho)\right)_C$ were received by Natanson [8]. In particular, he solved the Kolmogorov–Nikolsky problem on the classes $W^1_\infty$ for the Poisson integral. Namely, he proved the equality

$$\mathcal{E}\left(W^1_\infty; P(\rho)\right)_C = \frac{2}{\pi}\left(1-\rho\right)\left|\ln\left(1-\rho\right)\right| + O\left(1-\rho\right), \quad \rho \to 1-. \tag{9}$$

Timan [12] obtained the exact values of approximation characteristics $\mathcal{E}\left(W^r; P(\rho)\right)_C$ as $0 < \rho < 1$:

$$\mathcal{E}\left(W^r_\infty; P(0)\right)_C = K_r;$$

$$\mathcal{E}\left(W^1_\infty; P(\rho)\right)_C = \frac{2}{\pi}(1-\rho)\ln\frac{1}{1-\rho} + \varepsilon_\rho, \tag{10}$$

$$\varepsilon_\rho = \frac{2}{\pi}\int_0^{1-\rho}\left\{\frac{1}{1-t}\ln\frac{2-t}{t^t} + 1\right\}dt.$$

The equalities (9) and (10) as $\rho \to 1-$ allow us to write down constants, corresponding to the asymptotic term of the least order of the smallness.

In the paper of Shtark [10] the complete asymptotic expansion was obtained for the characteristics $\mathcal{E}\left(W^1_\infty; P\left(\rho\right)\right)_C$. It enables us to write down constants of an arbitrary order of smallness. Namely,

$$\mathcal{E}\left(W^1_\infty; P\left(\rho\right)\right)_C \cong \frac{2}{\pi}\sum_{k=1}^{\infty}\left\{\alpha_k\left(1-\rho\right)^k\ln\frac{1}{1-\rho} + \beta_k\left(1-\rho\right)^k\right\}, \tag{11}$$

where

$$\alpha_k = \frac{1}{k}, \quad \beta_k = \frac{1}{k}\Big(\frac{1}{k} + \ln 2 - \sum_{i=1}^{k-1}\frac{2^{-i}}{i}\Big).$$

In the work of Zhyhallo and Kharkevych [14] the complete asymptotic expansions were established for the quantity $\mathcal{E}(W^r_\infty; P(\rho))_C$ in the terms of $(1-\rho)$ as $\rho \to 1-$.

Later, approximation properties of the Poisson integrals were investigated on other functional classes. In particular, we note the works [1]–[7], [13], [15]–[17].

We will pay more attention to the work of Baskakov [1]. Note, that in the works [10] and [14] the complete asymptotic expansions were found in the terms of $(1-\rho)$ as $\rho \to 1-$. And in the paper [1], there were obtained the analogical expansions for the quantities $\mathcal{E}\left(W^1_\infty; P_\delta\right)_C$, but in terms of $1/\delta$ as $\delta \to \infty$:

$$\begin{aligned}\mathcal{E}\left(W^1_\infty; P_\delta\right)_C = \frac{2}{\pi}\frac{1}{\delta}\ln\delta + \frac{1}{\delta}\left[\frac{2\ln\pi}{\pi} + \frac{2}{\pi}\int\limits_\pi^\infty \frac{(t)_{2\pi}}{t^2}dt\right] \\ + \frac{2}{\pi\delta}\sum_{k=1}^\infty (-1)^k\left[\int\limits_\pi^\infty \frac{(t)_{2\pi}}{t^{2(k+1)}}dt - \frac{1}{2k\pi^{2k}}\right]\frac{1}{\delta^{2k}},\end{aligned} \tag{12}$$

where $(f(t))_{2\pi}$ is an even $2\pi$-periodic extension of the function $f(t)$, $0 \le t \le \pi$.

In this paper we continue and generalise the research of Baskakov. Our aim is to obtain the complete asymptotic expansions for the values $\mathcal{E}\left(W^r_\infty; P_\delta\right)_C$ and $\mathcal{E}\left(W^r_1; P_\delta\right)_1$ in terms of $1/\delta$ as $\delta \to \infty$ for an arbitrary $r \in \mathbb{N}$. This will allow us to calculate the Kolmogorov–Nikolsky constants of an arbitrary order of smallness.

Following theorems take place.

**Theorem 1.** *If $r = 2l-1$, $l \in \mathbb{N}$, then the following complete asymptotic expansion holds:*

$$\mathcal{E}\left(W^r_\infty; P_\delta\right)_C = \mathcal{E}\left(W^r_1; P_\delta\right)_1 \cong \frac{2}{\pi}\left(\frac{1}{r!}\frac{1}{\delta^r}\ln\delta + \sum_{k=1}^\infty \beta^r_k\frac{1}{\delta^k}\right), \quad \delta\to\infty, \tag{13}$$

*where*

$$\beta^r_k = \begin{cases} \frac{(-1)^{k-1}}{k!}\varphi_{r-k}, & k<r, \\ \frac{1}{r!}\left(\ln 2 + \sum_{i=1}^r \frac{1}{i}\right), & k=r, \\ \frac{(-1)^{k-1}}{k!}\sigma_{k-r}, & k>r, \end{cases} \tag{14}$$

$$\sigma_j = \frac{1}{2^{j-1}j!}\sum_{i=1}^j (2i-1)^j a_i^{j+1} - \frac{2^j(j-1)!}{(2j)!}\sum_{i=0}^{j-1}(-1)^i C^i_j (j-i)^{2j},$$

$$a^j_i = \begin{cases} 1, & i=1, \quad i=j-1, \\ a^{j-1}_i(2i-1) + a^{j-1}_{i-1}\left(2\left(j-i\right)-1\right), & 1<i<j-1, \end{cases} \tag{15}$$

$$\varphi_j = \begin{cases} \frac{\pi}{2}K_j, & j=2m-1, \\ \frac{\pi}{2}\widetilde{K}_j, & j=2m, \end{cases} \qquad m\in\mathbb{N}.$$

*Proof.* Using Theorem 4 from the article of Pych [9] for the Poisson integrals, we obtain

$$\mathcal{E}\left(W^r_\infty; P_\delta\right)_C = \mathcal{E}\left(W^r_1; P_\delta\right)_1, \quad r \in \mathbb{N}. \tag{16}$$

Therefore, it is sufficient to consider the case of the uniform metric.

Timan [12] obtained the following equalities:

$$\mathcal{E}\left(W^r_\infty; P_\delta\right)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1-e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}} = \frac{2}{\pi}\varphi_r\left(\delta\right), \quad r = 2l-1, \ l \in \mathbb{N}, \tag{17}$$

where

$$\varphi_r\left(\delta\right) = \int\limits_0^{\frac{1}{\delta}}\int\limits_{t_r}^{\infty}\ldots\int\limits_{t_2}^{\infty}\ln\frac{1+e^{-t_1}}{1-e^{-t_1}}dt_1dt_2\ldots dt_r$$

and

$$\varphi_n(0) = \varphi_n := \begin{cases} \frac{\pi}{2}K_n, & n = 2l-1, \\ \frac{\pi}{2}\widetilde{K}_n, & n = 2l, \end{cases} \qquad l \in \mathbb{N}.$$

We find the complete asymptotic expansions of the functions $\varphi_r(\delta)$ in the terms $1/\delta$ as $\delta \to \infty$.

Let

$$\varphi_r(\delta) = \alpha^r_1\frac{1}{\delta}\ln\delta + o\left(\frac{1}{\delta}\ln\delta\right),$$

then

$$\alpha^r_1 = \lim_{\delta\to\infty}\frac{\varphi_r\left(\delta\right)}{\frac{1}{\delta}\ln\delta} = -\lim_{\frac{1}{\delta}\to 0}\frac{\varphi_r(\delta)}{\frac{1}{\delta}\ln\frac{1}{\delta}}.$$

In the case

$$\varphi_r(\delta) = \alpha^r_1\frac{1}{\delta}\ln\delta + \beta^r_1\frac{1}{\delta} + o\left(\frac{1}{\delta}\right)$$

we obtain

$$\beta^r_1 = \lim_{\delta\to\infty}\frac{\varphi_r(\delta) - \alpha^r_1\frac{1}{\delta}\ln\delta}{\frac{1}{\delta}} = \lim_{\frac{1}{\delta}\to 0}\frac{\varphi_r(\delta) + \alpha^r_1\frac{1}{\delta}\ln\frac{1}{\delta}}{\frac{1}{\delta}}.$$

Thus, the possibility to write down the asymptotic expansion of functions $\varphi_r\left(\delta\right)$ in the form

$$\varphi_r\left(\delta\right) \cong \sum_{k=1}^{\infty}\left\{\alpha^r_k\frac{1}{\delta^k}\ln\delta + \beta^r_k\frac{1}{\delta^k}\right\} \tag{18}$$

is equivalent to the fact that the coefficients $\alpha^r_k$ and $\beta^r_k$ are connected with the function $\varphi_r\left(\delta\right)$ through the equalities

$$\alpha^r_k = \lim_{\delta\to\infty}\frac{1}{\frac{1}{\delta^k}\ln\delta}\left\{\varphi_r\left(\delta\right) - \sum_{j=1}^{k-1}\left(\alpha^r_j\frac{1}{\delta^j}\ln\delta + \beta^r_j\frac{1}{\delta^j}\right)\right\}, \tag{19}$$

$$\beta_k^r = \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^k}} \left\{ \varphi_r(\delta) - \alpha_k^r \frac{1}{\delta^k} \ln\delta - \sum_{j=1}^{k-1} \left( \alpha_j^r \frac{1}{\delta^j} \ln\delta + \beta_j^r \frac{1}{\delta^j} \right) \right\}. \qquad (20)$$

To check the conditions (7) and (8), it is necessary to assume that

$$g_{2k-1} = \alpha_k^r \frac{1}{\delta^k} \ln\delta, \quad g_{2k} = \beta_k^r \frac{1}{\delta^k}.$$

Using the L'Hospital rule $k$ times, we obtain that if $k=1$ and $r>1$, then

$$\alpha_1^r = \lim_{\delta\to\infty} \frac{\varphi_r(\delta)}{\frac{1}{\delta}\ln\delta}$$

$$= \lim_{\frac{1}{\delta}\to 0} \frac{-1}{1+\ln\frac{1}{\delta}} \int_{\frac{1}{\delta}}^{\infty} \int_{t_{r-1}}^{\infty} \ldots \int_{t_2}^{\infty} \ln\frac{1+e^{-t_1}}{1-e^{-t_1}} dt_1\, dt_2 \ldots dt_{r-1} = 0,$$

$$\beta_1^r = \lim_{\delta\to\infty} \frac{\varphi_r(\delta)}{\frac{1}{\delta}}$$

$$= \lim_{\frac{1}{\delta}\to 0} \int_{\frac{1}{\delta}}^{\infty} \int_{t_{r-1}}^{\infty} \ldots \int_{t_2}^{\infty} \ln\frac{1+e^{-t_1}}{1-e^{-t_1}} dt_1\, dt_2 \ldots dt_{r-1} = \varphi_{r-1}.$$

If $k=2$ and $r>2$, then

$$\alpha_2^r = \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^2}\ln\delta} \left\{ \varphi_r(\delta) - \varphi_{r-1}\frac{1}{\delta} \right\}$$

$$= \lim_{\frac{1}{\delta}\to 0} \frac{-1}{2\ln\frac{1}{\delta}+3} \int_{\frac{1}{\delta}}^{\infty} \int_{t_{r-2}}^{\infty} \ldots \int_{t_2}^{\infty} \ln\frac{1+e^{-t_1}}{1-e^{-t_1}} dt_1\, dt_2 \ldots dt_{r-2} = 0,$$

$$\beta_2^r = \lim_{\delta\to\infty} \frac{\varphi_r(\delta) - \varphi_{r-1}\frac{1}{\delta}}{\frac{1}{\delta^2}}$$

$$= -\lim_{\frac{1}{\delta}\to 0} \frac{1}{2} \int_{\frac{1}{\delta}}^{\infty} \int_{t_{r-2}}^{\infty} \ldots \int_{t_2}^{\infty} \ln\frac{1+e^{-t_1}}{1-e^{-t_1}} dt_1\, dt_2 \ldots dt_{r-2} = -\frac{1}{2}\varphi_{r-2}.$$

In the case of $k \le r-1$, using the obvious equalities

$$\frac{d^k \varphi_r(\delta)}{d\left(\frac{1}{\delta}\right)^k} = (-1)^{k-1} \int_{\frac{1}{\delta}}^{\infty} \int_{t_{r-k}}^{\infty} \ldots \int_{t_2}^{\infty} \ln\frac{1+e^{-t_1}}{1-e^{-t_1}} dt_1\, dt_2 \ldots dt_{r-k},$$

$$\frac{d^k}{d\left(\frac{1}{\delta}\right)^k} \Big( \frac{1}{\delta^k} \ln\frac{1}{\delta} \Big) = k! \Big( \ln\frac{1}{\delta} + \sum_{i=1}^{k} \frac{1}{i} \Big), \qquad (21)$$

we get

$$\begin{aligned}
\alpha_k^r &= \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^k}\ln\delta}\left\{\varphi_r(\delta) - \sum_{i=1}^{k-1}\beta_i^r \frac{1}{\delta^i}\right\} \\
&= \lim_{\frac{1}{\delta}\to 0} \frac{(-1)^k}{k!\big(\ln\frac{1}{\delta} + \sum\limits_{i=1}^{k}\frac{1}{i}\big)} \int\limits_{\frac{1}{\delta}}^{\infty}\int\limits_{t_{r-k}}^{\infty}\dots\int\limits_{t_2}^{\infty} \ln\frac{1+e^{-t_1}}{1-e^{-t_1}} dt_1\, dt_2 \dots dt_{r-k} \\
&= 0,
\end{aligned} \tag{22}$$

$$\begin{aligned}
\beta_k^r &= \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^k}}\left\{\varphi_r(\delta) - \sum_{i=1}^{k-1}\beta_i^r \frac{1}{\delta^i}\right\} \\
&= \lim_{\frac{1}{\delta}\to 0} \frac{1}{k!}(-1)^{k-1} \int\limits_{\frac{1}{\delta}}^{\infty}\int\limits_{t_{r-k}}^{\infty}\dots\int\limits_{t_2}^{\infty} \ln\frac{1+e^{-t_1}}{1-e^{-t_1}} dt_1\, dt_2 \dots dt_{r-k} \\
&= \frac{(-1)^{k-1}}{k!}\varphi_{r-k}.
\end{aligned} \tag{23}$$

If $k = r$, then we have

$$\alpha_r^r = \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^r}\ln\delta}\left\{\varphi_r(\delta) - \sum_{i=1}^{r-1}\beta_i^r \frac{1}{\delta^i}\right\} = \lim_{\frac{1}{\delta}\to 0} \frac{-1}{r!\big(\ln\frac{1}{\delta} + \sum\limits_{i=1}^{r}\frac{1}{i}\big)} \frac{d^r\varphi_r(\delta)}{d\left(\frac{1}{\delta}\right)^r},$$

$$\begin{aligned}
\beta_r^r &= \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^r}}\left\{\varphi_r(\delta) - \alpha_r^r\frac{1}{\delta^r}\ln\delta - \sum_{i=1}^{r-1}\beta_i^r \frac{1}{\delta^i}\right\} \\
&= \lim_{\frac{1}{\delta}\to 0} \frac{1}{r!}\left\{\frac{d^r\varphi_r(\delta)}{d\left(\frac{1}{\delta}\right)^r} + \alpha_r^r r!\left(\ln\frac{1}{\delta} + \sum_{i=1}^{r}\frac{1}{i}\right)\right\}.
\end{aligned}$$

So, considering the validity of the equality

$$\frac{d^r\varphi_r(\delta)}{d\left(\frac{1}{\delta}\right)^r} = \ln(1+e^{-\frac{1}{\delta}}) - \ln(1-e^{-\frac{1}{\delta}}),$$

we get

$$\alpha_r^r = \lim_{\frac{1}{\delta}\to 0} \frac{\ln(1-e^{-\frac{1}{\delta}}) - \ln(1+e^{-\frac{1}{\delta}})}{r!\left(\ln\frac{1}{\delta} + \sum\limits_{i=1}^{r}\frac{1}{i}\right)} = \frac{1}{r!}, \tag{24}$$

$$\begin{aligned}\beta_r^r &= \lim_{\frac{1}{\delta}\to 0}\frac{1}{r!}\left(\ln\left(1+e^{-\frac{1}{\delta}}\right)-\ln\left(1-e^{-\frac{1}{\delta}}\right)+\ln\frac{1}{\delta}+\sum_{i=1}^{r}\frac{1}{i}\right)\\ &= \frac{1}{r!}\left(\ln 2+\sum_{i=1}^{r}\frac{1}{i}\right).\end{aligned} \tag{25}$$

Now we study the case when $k>r$. It is possible to show that

$$\frac{d^k\varphi_r(\delta)}{d\left(\frac{1}{\delta}\right)^k}=\frac{(-1)^{k-1}2\sum\limits_{i=1}^{k-r}a_i^{k-r+1}e^{(2i-1)\frac{1}{\delta}}}{\left(e^{\frac{2}{\delta}}-1\right)^{k-r}}, \tag{26}$$

where $a_i^j$ are defined by correlation (15), and

$$\frac{d^k}{d\left(\frac{1}{\delta}\right)^k}\left(\frac{1}{\delta^\mu}\ln\frac{1}{\delta}\right)=(-1)^{k-\mu-1}\mu!(k-\mu-1)!\delta^{k-\mu},\quad k>\mu. \tag{27}$$

If $k=r+1$, then

$$\begin{aligned}\alpha_{r+1}^r &= \lim_{\delta\to\infty}\frac{1}{\frac{1}{\delta^{r+1}}\ln\delta}\left\{\varphi_r(\delta)-\alpha_r^r\frac{1}{\delta^r}\ln\delta-\sum_{i=1}^{r}\beta_i^r\frac{1}{\delta^i}\right\}\\ &= \lim_{\frac{1}{\delta}\to 0}\frac{-1}{(r+1)!\left(\ln\frac{1}{\delta}+\sum\limits_{i=1}^{r+1}\frac{1}{i}\right)}\left\{\frac{d^{r+1}\varphi_r(\delta)}{d\left(\frac{1}{\delta}\right)^{r+1}}+\alpha_r^r\frac{d^{r+1}\left(\frac{1}{\delta^r}\ln\frac{1}{\delta}\right)}{d\left(\frac{1}{\delta}\right)^{r+1}}\right\}\\ &= \lim_{\frac{1}{\delta}\to 0}\frac{-1}{(r+1)!\left(\ln\frac{1}{\delta}+\sum\limits_{i=1}^{r+1}\frac{1}{i}\right)}\left(-\frac{2e^{\frac{1}{\delta}}}{e^{\frac{2}{\delta}}-1}+\delta\right)=0.\end{aligned}$$

Further, using similar transformations, we obtain

$$\beta_{r+1}^r=\lim_{\frac{1}{\delta}\to 0}\frac{1}{(r+1)!}\left(-\frac{2e^{\frac{1}{\delta}}}{e^{\frac{2}{\delta}}-1}+\delta\right)=0.$$

From the equalities (26), (27), and the formulas (19) and (20) we get, that if $k>r$, then the following equalities hold:

$$\begin{aligned}\alpha_k^r &= \lim_{\delta\to\infty}\frac{1}{\frac{1}{\delta^k}\ln\delta}\left\{\varphi_r(\delta)-\alpha_r^r\frac{1}{\delta^r}\ln\delta-\sum_{i=1}^{k-1}\beta_i^r\frac{1}{\delta^i}\right\}\\ &= \lim_{\frac{1}{\delta}\to 0}\frac{-1}{k!\left(\ln\frac{1}{\delta}+\sum\limits_{i=1}^{k}\frac{1}{i}\right)}\left\{\frac{d^k\varphi_r(\delta)}{d\left(\frac{1}{\delta}\right)^k}+\alpha_r^r\frac{d^k}{d\left(\frac{1}{\delta}\right)^k}\left(\frac{1}{\delta^r}\ln\frac{1}{\delta}\right)\right\}\end{aligned}$$

$$= \lim_{\frac{1}{\delta}\to 0} \frac{(-1)^k}{k!\left(\ln\frac{1}{\delta} + \sum\limits_{i=1}^{k}\frac{1}{i}\right)} A_{k-r}(\delta),$$

$$\begin{aligned}
\beta_k^r &= \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^k}} \left\{ \varphi_r(\delta) - \alpha_r^r \frac{1}{\delta^r}\ln\delta - \sum_{i=1}^{k-1} \beta_i^r \frac{1}{\delta^i} \right\} \\
&= \lim_{\frac{1}{\delta}\to 0} \frac{1}{k!} \left\{ \frac{d^k \varphi_r(\delta)}{d\left(\frac{1}{\delta}\right)^k} + \alpha_r^r \frac{d^k}{d\left(\frac{1}{\delta}\right)^k} \left( \frac{1}{\delta^r} \ln\frac{1}{\delta} \right) \right\} \\
&= \lim_{\frac{1}{\delta}\to 0} \frac{(-1)^{k-1}}{k!} A_{k-r}(\delta),
\end{aligned}$$

where

$$A_j(\delta) := \frac{2\sum\limits_{i=1}^{j} a_i^{j+1} e^{(2i-1)\frac{1}{\delta}}}{(e^{\frac{2}{\delta}}-1)^j} - \frac{(j-1)!}{\frac{1}{\delta^j}}.$$

Obviously

$$\sigma_j = \lim_{\frac{1}{\delta}\to 0} A_j(\delta) = \frac{1}{2^j} \lim_{\frac{1}{\delta}\to 0} \frac{2\frac{1}{\delta^j}\sum\limits_{i=1}^{j} a_i^{j+1} e^{(2i-1)\frac{1}{\delta}} - (j-1)!\left(e^{\frac{2}{\delta}}-1\right)^j}{\frac{1}{\delta^{2j}}}.$$

Further, using the L'Hospital rule $2j$ times, we obtain

$$\begin{aligned}
\sigma_j &= \frac{1}{2^j} \lim_{\frac{1}{\delta}\to 0} \left( \frac{2\sum\limits_{k=0}^{j} C_{2j}^k \frac{j!}{(j-k)!}\frac{1}{\delta^{j-k}} \sum\limits_{i=1}^{j} a_i^{j+1}(2i-1)^{2j-k} e^{(2i-1)\frac{1}{\delta}}}{(2j)!} \right. \\
&\qquad \left. - \frac{(j-1)!\sum\limits_{i=0}^{j-1} C_j^i (-1)^i 2^{2j} (j-i)^{2j} e^{\frac{2(j-i)}{\delta}}}{(2j)!} \right) \\
&= \frac{1}{2^j} \frac{2C_{2j}^j j! \sum\limits_{i=1}^{j} a_i^{j+1}(2i-1)^j - (j-1)!\sum\limits_{i=0}^{j-1} C_j^i (-1)^i 2^{2j}(j-i)^{2j}}{(2j)!} \\
&= \frac{1}{2^{j-1}j!} \sum_{i=1}^{j} (2i-1)^j a_i^{j+1} - \frac{2^j(j-1)!}{(2j)!} \sum_{i=0}^{j-1} (-1)^i C_j^i (j-i)^{2j},
\end{aligned}$$

where $a_i^j$ are defined by the correlation (15).

In the case $k > r$ we get

$$\alpha_k^r = 0, \tag{28}$$

$$\beta_k^r = \frac{(-1)^{k-1}}{k!}\sigma_{k-r}. \tag{29}$$

From the formulas (18), (22)–(25), (28), and (29) we get the complete asymptotic expansion of the function $\varphi_r(\delta)$, $r = 2l-1$, $l \in \mathbb{N}$,

$$\varphi_r(\delta) \cong \frac{1}{r!}\frac{1}{\delta^r}\ln\delta + \sum_{k=1}^{\infty}\beta_k^r\frac{1}{\delta^k}, \tag{30}$$

where $\beta_k^r$ are defined by the formula (14).

From the formulas (17) and (30) we obtain the equality (13). Theorem 1 is proven. $\square$

**Remark 1.** By virtue of Theorem 1, we can write the following complete asymptotic expansion in the case $r = 1$:

$$\mathcal{E}\left(W_\infty^1; P_\delta\right)_C \cong \frac{2}{\pi}\left(\frac{1}{\delta}\ln\delta + \sum_{k=1}^{\infty}\beta_k^1\frac{1}{\delta^k}\right). \tag{31}$$

It is necessary to notice that the results, written down by formulas (12) and (31), are different by the form. However it is possible to show the equivalence of coefficients with respect to the terms $1/\delta$ in the mentioned formulas:

$$\beta_1^1 = \frac{2\ln\pi}{\pi} + \frac{2}{\pi}\int_\pi^\infty \frac{(t)_{2\pi}}{t^2}dt = 1 + \ln 2, \quad \beta_2^1 = 0,$$

$$\beta_3^1 = -\frac{2}{\pi}\left(\int_\pi^\infty \frac{(t)_{2\pi}}{t^4}dt - \frac{1}{2\pi^2}\right) = \frac{1}{(3!)^2}, \quad \beta_4^1 = 0,$$

$$\beta_5^1 = \frac{2}{\pi}\left(\int_\pi^\infty \frac{(t)_{2\pi}}{t^6}dt - \frac{1}{2\pi^4}\right) = -\frac{14}{(5!)^2}, \quad \beta_6^1 = 0,$$

$$\beta_7^1 = \frac{2}{\pi}\left(\int_\pi^\infty \frac{(t)_{2\pi}}{t^8}dt - \frac{1}{2\pi^6}\right) = \frac{1240}{(7!)^2}, \quad \beta_8^1 = 0,$$

etc.

**Theorem 2.** *If $r = 2l$, $l \in \mathbb{N}$, then the complete asymptotic expansion*

$$\mathcal{E}\left(W_\infty^r; P_\delta\right)_C = \mathcal{E}\left(W_1^r; P_\delta\right)_1 \cong \frac{4}{\pi}\sum_{k=1}^{\infty}\gamma_k^r\frac{1}{\delta^k}, \quad \delta \to \infty, \tag{32}$$

*holds, where*

$$\gamma_k^r = \begin{cases} \frac{(-1)^{k-1}}{k!}\psi_{r-k}, & k < r, \\ -\frac{\pi}{4r!}, & k = r, \\ \frac{1}{k!}\tau_{k-r}, & k > r, \end{cases} \quad k,\ r \in \mathbb{N}, \tag{33}$$

$$\tau_j = \begin{cases} 0, & j = 2l, \\ \frac{1}{2^j}\sum\limits_{i=1}^{j}(-1)^{i-1}a_i^{j+1}, & j = 2l-1, \end{cases} \quad l \in \mathbb{N},$$

*the coefficients $a_i^j$ are defined by the formula* (15)*, and*

$$\psi_j = \begin{cases} \frac{\pi}{4}\widetilde{K}_j, & j = 2m-1, \\ \frac{\pi}{4}K_j, & j = 2m, \end{cases} \quad m \in \mathbb{N}.$$

*Proof.* According to the equality (16), it is sufficient to prove Theorem 2 only for the case of the uniform metric. In Timan's article [12] it was shown that

$$\mathcal{E}\left(W_\infty^r; P_\delta\right)_C = \frac{4}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1-e^{-\frac{2k+1}{\delta}}}{(2k+1)^{r+1}} = \frac{4}{\pi}\psi_r\left(\delta\right), \quad r = 2l,\ l \in \mathbb{N}, \tag{34}$$

where

$$\psi_r\left(\delta\right) = \int_0^{\frac{1}{\delta}}\int_{t_n}^{\infty}\ldots\int_{t_2}^{\infty}\arctan e^{-t_1}dt_1\ldots dt_n$$

and

$$\psi_n(0) = \psi_n := \begin{cases} \frac{\pi}{4}\widetilde{K}_n, & n = 2l-1, \\ \frac{\pi}{4}K_n, & n = 2l, \end{cases} \quad l \in \mathbb{N}.$$

Let us find complete asymptotic expansion of the function $\psi_r\left(\delta\right)$ on degrees of $1/\delta$ as $\delta \to \infty$.

In the asymptotic expansion of the function $\psi_r\left(\delta\right)$ of the form

$$\psi_r\left(\delta\right) \cong \sum_{k=1}^{\infty}\gamma_k^r\frac{1}{\delta^k} \tag{35}$$

the coefficients $\gamma_k^r$ are connected with the function $\psi_r\left(\delta\right)$ by the correlations

$$\gamma_k^r = \lim_{\delta\to\infty}\frac{1}{\frac{1}{\delta^k}}\left\{\psi_r\left(\delta\right) - \sum_{i=1}^{k-1}\gamma_i^r\frac{1}{\delta^i}\right\}.$$

Applying $k$ times the L'Hospital rule in the case $k \le r-1$, and taking into consideration the equality

$$\frac{d^k\psi_r\left(\delta\right)}{d\left(\frac{1}{\delta}\right)^k} = (-1)^{k-1}\int_{\frac{1}{\delta}}^{\infty}\int_{t_{r-k}}^{\infty}\ldots\int_{t_2}^{\infty}\arctan e^{-t_1}\,dt_1\,dt_2\ldots dt_{r-k},$$

we get

$$\gamma_k^r = \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^k}} \left\{ \psi_r(\delta) - \sum_{i=1}^{k-1} \gamma_i^r \frac{1}{\delta^i} \right\}$$
$$= \frac{(-1)^{k-1}}{k!} \lim_{\frac{1}{\delta}\to 0} \int_{\frac{1}{\delta}}^{\infty} \int_{t_{r-k}}^{\infty} \dots \int_{t_2}^{\infty} \arctan e^{-t_1}\, dt_1\, dt_2 \dots dt_{r-k} \tag{36}$$
$$= \frac{(-1)^{k-1}}{k!} \psi_{r-k}.$$

If $k = r$, then, using the fact that

$$\frac{d^r \psi_r(\delta)}{d\left(\frac{1}{\delta}\right)^r} = -\arctan e^{-\frac{1}{\delta}},$$

we obtain the equalities

$$\gamma_r^r = \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^r}} \left\{ \psi_r(\delta) - \sum_{i=1}^{r-1} \gamma_i^r \frac{1}{\delta^i} \right\} = -\frac{1}{r!} \lim_{\frac{1}{\delta}\to 0} \arctan e^{-\frac{1}{\delta}} = -\frac{\pi}{4r!}. \tag{37}$$

At last, we consider the case $k > r$. In this case

$$\frac{d^k \psi_r(\delta)}{d\left(\frac{1}{\delta}\right)^k} = \frac{\sum\limits_{i=1}^{k-r} (-1)^{i-1} a_i^{k-r+1} e^{(2i-1)\frac{1}{\delta}}}{(e^{\frac{2}{\delta}}+1)^{k-r}}.$$

Taking into account the fact that $a_i^{j+1} = a_{j+1-i}^{j+1}$, $1 \le i \le j$, we have

$$\tau_j = \lim_{\frac{1}{\delta}\to 0} \frac{\sum\limits_{i=1}^{j} (-1)^{i-1} a_i^{j+1} e^{(2i-1)\frac{1}{\delta}}}{(e^{\frac{2}{\delta}}+1)^j} = \begin{cases} 0, & j = 2l, \\ \frac{1}{2^j} \sum\limits_{i=1}^{j} (-1)^{i-1} a_i^{j+1}, & j = 2l-1. \end{cases}$$

So, we obtain

$$\begin{aligned} \gamma_k^r &= \lim_{\delta\to\infty} \frac{1}{\frac{1}{\delta^k}} \left\{ \psi_r(\delta) - \sum_{i=1}^{k-1} \gamma_i^r \frac{1}{\delta^i} \right\} = \frac{1}{k!} \lim_{\frac{1}{\delta}\to 0} \frac{d^k \psi_r(\delta)}{d\left(\frac{1}{\delta}\right)^k} \\ &= \frac{1}{k!} \lim_{\frac{1}{\delta}\to 0} \frac{\sum\limits_{i=1}^{k-r} (-1)^{i-1} a_i^{k-r+1} e^{(2i-1)\frac{1}{\delta}}}{(e^{\frac{2}{\delta}}+1)^{k-r}} = \frac{1}{k!} \tau_{k-r}. \end{aligned} \tag{38}$$

From the correlations (35)–(38) for the function $\psi_r(\delta)$ we make the conclusion, that the following complete asymptotic expansion holds:

$$\psi_r(\delta) \cong \sum_{k=1}^{\infty} \gamma_k^r \frac{1}{\delta^k}. \tag{39}$$

From the formulas (34) and (39), we get the equality (32). Theorem 2 is proven. □

Let us notice, that in an explicit form the asymptotic expansion of the quantity $\mathcal{E}\left(W^r_\infty; P_\delta\right)_C$ in the terms $1/\delta$ were written down only for the cases $r = 1, 2, 3$ by Baskakov (the case $r = 1$ was represented by the formula (12)).

In contrast to the results of Baskakov, Theorems 1 and 2 contain the complete asymptotic expansions of the quantities $\mathcal{E}\left(W^r_\infty; P_\delta\right)_C$ and $\mathcal{E}\left(W^r_1; P_\delta\right)_1$ in an explicit form for all powers of smoothness $r \in \mathbb{N}$.

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