On a class of vector-valued entire Dirichlet series in \( n \) variables

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Abstract. We consider a class \( F \) of entire Dirichlet series in \( n \) variables, whose coefficients belong to a commutative Banach algebra \( E \). With a well defined norm, \( F \) is proved to be a Banach algebra with identity. Further results on quasi-invertibility, spectrum and continuous linear functionals are proved for elements belonging to \( F \).

1. Introduction

Let \( E \) be a commutative Banach algebra with the identity such that all non-null elements of \( E \) are invertible. In this case \( E \) is also a field. Let

\[
    f(s_1, s_2, \ldots, s_n) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} a_{m_1,m_2,\ldots,m_n} e^{(\lambda_{m_1} s_1 + \lambda_{m_2} s_2 + \cdots + \lambda_{m_n} s_n)}
\]

be an \( n \)-tuple Dirichlet series, where \( a_{m_1,m_2,\ldots,m_n} \in E \), \( s_j = \sigma_j + it_j \), \( j = 1, 2, \ldots, n \), and

\[
    0 < \lambda_{p_1} < \lambda_{p_2} < \cdots < \lambda_{p_k} \to \infty \text{ as } k \to \infty, \text{ for } p = 1, 2, \ldots, n.
\]

To simplify the form of \( n \)-tuple Dirichlet series, we use the following notation:

\[
    s = (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n,
\]

\[
    m = (m_1, m_2, \ldots, m_n) \in \mathbb{C}^n,
\]

\[
    \lambda_{m_n} = (\lambda_{m_1}, \lambda_{m_2}, \ldots, \lambda_{m_n}) \in \mathbb{R}^n.
\]

We further define

\[
    \lambda_{m_n} s = \lambda_{m_1} s_1 + \lambda_{m_2} s_2 + \cdots + \lambda_{m_n} s_n,
\]

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Thus, the series (1.1) can be written as

$$f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{nmn} s}. \quad (1.2)$$

Janusauskas [1] showed that if there exists a tuple $p > 0 = (0, 0, \ldots, 0)$ such that

$$\limsup_{|m| \to \infty} \sum_{k=1}^{\infty} \frac{\log m_k}{p \lambda_{nmn}} = 0, \quad (1.3)$$

then the domain of absolute convergence of (1.2) coincides with its domain of convergence. Sarkar [6] proved that a necessary and sufficient condition for the series (1.2) with $a_m \in \mathbb{C}$ and satisfying (1.3) to be entire, i.e., to converge in the whole complex plane, is

$$\lim_{|m| \to \infty} \frac{\log |a_m|}{\lambda_{nmn}} = -\infty. \quad (1.4)$$

Liang and Gao [5] investigated the convergence and growth of $n$-tuple Dirichlet series and thus established an equivalence relation between order and the coefficients. Vaish [7] proved a necessary and sufficient condition so that Goldberg order of multiple Dirichlet series defining an entire function remained unaltered under rearrangements of the coefficients of the series. Kumar and Manocha in [2] generalised the condition of weighted norm for a Dirichlet series in one variable. In the present paper, we extend some results from [2] and also prove some new results for a Dirichlet series in $n$ variables.

2. Basic results

In this section, some basic results are proved which are required to prove our main results.

**Lemma 1.** The following conditions are equivalent.

(a) $\limsup_{|m| \to \infty} \frac{\log (m_1 + m_2 + \cdots + m_n)}{\lambda_{1m1} + \lambda_{2m2} + \cdots + \lambda_{nmn}} = D < \infty$.

(b) $\limsup_{m_k \to \infty} \frac{\log m_k}{\lambda_{kmk}} = D_k < \infty, \quad k \in \mathbb{N} = \{1, 2, \ldots\}$.

(c) There exists $\alpha, \quad 0 < \alpha < \infty$, such that the series

$$\sum_{m_1, m_2, \ldots, m_n=1}^{\infty} e^{-\alpha(\lambda_{1m1} + \lambda_{2m2} + \cdots + \lambda_{nmn})} \quad (2.1)$$

converges.
Proof. (a)⇒(b). Let \( \epsilon > 0 \) be arbitrary. There exists \( M_0 = M_0(\epsilon) \) such that
\[
\log (m_1 + m_2 + \cdots + m_n) < (D + \epsilon) (\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n})
\]
for \( m_1 + m_2 + \cdots + m_n \geq M_0 \). According to the relation
\[
\sqrt{m_1m_2 \cdots m_n} \leq (m_1 + m_2 + \cdots + m_n),
\]
we get
\[
\frac{\log m_k}{\lambda_{km_k}} < 2(D + \epsilon) \frac{\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n}}{\lambda_{km_k}}, \quad 1 \leq k \leq n, \quad m_k \geq M,
\]
where \( M = M(\epsilon) \).

(b)⇒(c). For arbitrary numbers \( \epsilon_k > 0, \quad 1 \leq k \leq n \), we may choose \( M_k(\epsilon_k), \quad 1 \leq k \leq n \), such that
\[
\log m_k < (D_k + \epsilon_k) \lambda_{km_k}, \quad 1 \leq k \leq n, \quad m_k \geq M_k(\epsilon_k).
\]
Then the series (2.1) converges for \( \alpha > \max_{1 \leq k \leq n} (D_k + \epsilon_k) \).

(c)⇒(a). Let \( S \) be the sum of the series (2.1). There exists \( M_0 \in \mathbb{N} \) such that
\[
\sum_{m_1, m_2, \ldots, m_n} e^{-\alpha (\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n})} < S \quad \text{for} \quad m_1, m_2, \ldots, m_n \geq M_0.
\]
Hence
\[
\sum_{m_1, m_2, \ldots, m_n} e^{-\alpha (\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n})} \leq S < \infty.
\]
Thus
\[
\limsup_{m_1 + m_2 + \cdots + m_n \to \infty} \frac{\log (m_1 + m_2 + \cdots + m_n)}{\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n}} \leq \alpha.
\]

\( \square \)

Theorem 1. A necessary and sufficient condition for the series (1.1) satisfying (1.3) to be entire is that
\[
\limsup_{m_1 + m_2 + \cdots + m_n \to \infty} \frac{\log (\|a_{m_1, m_2, \ldots, m_n}\|)}{\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n}} = -\infty. \quad (2.2)
\]
Proof. Suppose that (1.1) defines an entire function. Then it converges absolutely for all \((s_1, s_2, \ldots, s_n)\). Thus, taking the points with coordinates \((q, q, \ldots, q)\), where \(q > 0\), it follows that

\[
\sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \|a_{m_1, m_2, \ldots, m_n}\| e^{(\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n})q} < \infty.
\]

Therefore,

\[
\|a_{m_1, m_2, \ldots, m_n}\| e^{(\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n})q} \leq T(q, q, \ldots, q) < 1
\]

for sufficiently large values of \(m_1 + m_2 + \cdots + m_n\). Hence

\[
\limsup_{m_1 + m_2 + \cdots + m_n \to \infty} \frac{\log \|a_{m_1, m_2, \ldots, m_n}\|}{\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n}} < -\sigma.
\]

Since \(q > 0\) is arbitrary, the “necessity” part of the theorem is proved.

Conversely, let (2.2) be satisfied. It is sufficient to prove that (1.1) converges for all \((s_1, s_2, \ldots, s_n)\). Let \(\sigma > 0\) be such that \(\Re s_1 < \sigma, \Re s_2 < \sigma, \ldots, \Re s_n < \sigma\). By Lemma 1 we can find \(\alpha > 0\) such that the series (2.1) converges. Now we fix \(N_0(\alpha)\) such that

\[
\frac{\log \|a_{m_1, m_2, \ldots, m_n}\|}{\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n}} < -\sigma - \alpha \quad \text{for} \quad m_1, m_2, \ldots, m_n \geq N_0(\alpha).
\]

Thus

\[
\|a_{m_1, m_2, \ldots, m_n}\| e^{(\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n})\sigma} < e^{-a(\lambda_{1m_1} + \lambda_{2m_2} + \cdots + \lambda_{nm_n})}.
\]

Since the series (2.1) converges, the series (1.1) converges absolutely for all \((s_1, s_2, \ldots, s_n)\). \(\square\)

From the above notations one gets

\[
\lim_{|m| \to \infty} \frac{\log \|a_m\|}{\|\lambda_{nm_n}\|} = -\infty. \quad (2.3)
\]

We denote by \(F\) the set of the series (1.2) satisfying (1.3) and for which the sequence of numbers

\[
|\lambda_{nm_n}| c_1 |\lambda_{nm_n}| (|m|!)^{c_2} \|a_m\|
\]

is bounded, where \(c_1, c_2 \geq 0\) and \(c_1, c_2\) are simultaneously not zero. Thus there exists a number \(G\) such that

\[
|\lambda_{nm_n}| c_1 |\lambda_{nm_n}| (|m|!)^{c_2} \|a_m\| < G,
\]

\[
c_1 \log |\lambda_{nm_n}| + c_2 \frac{\log (|m|!)}{|\lambda_{nm_n}|} + \log \|a_m\| \frac{\log |\lambda_{nm_n}|}{|\lambda_{nm_n}|} < \log G,
\]

\[
\frac{\log \|a_m\|}{|\lambda_{nm_n}|} < \left\{ c_1 \log |\lambda_{nm_n}| + c_2 \frac{\log (|m|!)}{|\lambda_{nm_n}|} + \log G \right\}.
\]
This implies (2.3). Hence by Theorem 1 every element of $F$ represents an entire function.

Define binary operations in $F$ as
\[
\begin{align*}
  f(s) + g(s) &= \sum_{m=1}^{\infty} (a_m + b_m) \ e^{\lambda_{nm}s}, \\
  \gamma f(s) &= \sum_{m=1}^{\infty} (\gamma a_m) \ e^{\lambda_{nm}s}, \quad \gamma \in E, \\
  f(s) \cdot g(s) &= \sum_{m=1}^{\infty} |\lambda_{nm}| c_1 |\lambda_{nm}| \ (|m|!)^c_2 a_m b_n \ e^{\lambda_{nm}s}.
\end{align*}
\]

The norm in $F$ is defined by
\[
\|f\| = \sup_{|m| \geq 1} |\lambda_{nm}| c_1 |\lambda_{nm}| \ (|m|!)^c_2 \|a_m\|.
\] (2.4)

It is not difficult to see that $F$ forms a linear space over the field $E$.

3. Main results

**Theorem 2.** $F$ is a commutative Banach algebra with identity over the field $E$.

**Proof.** In order to prove this theorem we need to show that $F$ is complete under the norm defined by (2.4). Let $\{f_{t_1}\}$ be a Cauchy sequence in $F$, where
\[
f_{t_1}(s) = \sum_{m=1}^{\infty} a_m^{(t_1)} \ e^{\lambda_{nm}s}.
\]

Then, for given $\epsilon > 0$, we can find $t \geq 1$ such that $\|f_{t_1} - f_{t_2}\| < \epsilon$ for $t_1, t_2 \geq t$, i.e.,
\[
\sup_{|m| \geq 1} |\lambda_{nm}| c_1 |\lambda_{nm}| \ (|m|!)^c_2 \|a_m^{(t_1)} - a_m^{(t_2)}\| < \epsilon, \quad t_1, t_2 \geq t.
\]

This shows that $\{a_m^{(t_1)}\}$ forms a Cauchy sequence in a Banach space $E$ for all values of $|m| \geq 1$. Hence
\[
\lim_{t_1 \to \infty} a_m^{(t_1)} = a_m, \quad |m| \geq 1.
\]

Letting $t_2 \to \infty$, we get
\[
\sup_{|m| \geq 1} |\lambda_{nm}| c_1 |\lambda_{nm}| \ (|m|!)^c_2 \|a_m - a_m\| < \epsilon, \quad t_1 \geq t.
\]

Thus $f_{t_1} \to f$ as $t_1 \to \infty$. Also
\[
\sup_{|m| \geq 1} |\lambda_{nm}| c_1 |\lambda_{nm}| \ (|m|!)^c_2 \|a_m\| \leq \sup_{|m| \geq 1} |\lambda_{nm}| c_1 |\lambda_{nm}| \ (|m|!)^c_2 \|a_m - a_m\|
\]
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The identity element in $F$ is

$$e(s) = \sum_{m=1}^{\infty} e|\lambda_{nm}|^{-c_1}|\lambda_{nm}|^{-c_2}e^{\lambda_{nm}s},$$

where $e$ is the identity element of $E$. Now, if $f, g \in F$, then

$$\|f \cdot g\| = \sup_{|m| \geq 1} |\lambda_{nm}|^{-c_1}|\lambda_{nm}|^{-c_2} \left\{ \left| \lambda_{nm} |^{-c_1}|\lambda_{nm}|^{-c_2} A_m b_m \right| \right\}$$

$$\leq \|f\| \|g\|.$$  

This proves the theorem.  

**Theorem 3.** The function (1.2) is invertible in $F$ if and only if the sequence

$$\left\{ \left| d_m |^{-c_1}|\lambda_{nm}|^{-c_2} \right| \right\}$$

(3.1)

is bounded, where $d_m$ is the inverse of $a_m$.

**Proof.** Let $f \in F$ be invertible and let $g(s) = \sum_{m=1}^{\infty} b_m e^{\lambda_{nm}s}$ be its inverse. Then $f(s) \cdot g(s) = e(s)$. Therefore,

$$|\lambda_{nm}|^{c_1}|\lambda_{nm}|^{-c_2} A_m b_m = e|\lambda_{nm}|^{-c_1}|\lambda_{nm}|^{-c_2},$$

which implies that

$$|\lambda_{nm}|^{c_1}|\lambda_{nm}|^{-c_2} \|b_m\| = \left\| e \left\{ |\lambda_{nm}|^{c_1}|\lambda_{nm}|^{-c_2} A_m \right\}^{-1} \right\|$$

or, equivalently,

$$|\lambda_{nm}|^{c_1}|\lambda_{nm}|^{-c_2} \|b_m\| = \left\| d_m |\lambda_{nm}|^{-c_1}|\lambda_{nm}|^{-c_2} \right\| .$$

Thus (3.1) is a bounded sequence since $g \in F$.

Conversely, suppose that the sequence (3.1) is bounded. Define $g(s)$ such that

$$g(s) = \sum_{m=1}^{\infty} e|\lambda_{nm}|^{-2c_1}|\lambda_{nm}|^{-2c_2} A_m^{-1} e^{\lambda_{nm}s}.$$

Then

$$f(s) \cdot g(s) = \sum_{m=1}^{\infty} e|\lambda_{nm}|^{-c_1}|\lambda_{nm}|^{-c_2} e^{\lambda_{nm}s} = e(s).$$

The proof is complete.
Definition 1. A function $g \in F$ is said to be quasi-inverse of $f \in F$ if $f(s) \ast g(s) = 0$, where

$$f(s) \ast g(s) = f(s) + g(s) + f(s) \cdot g(s).$$

Theorem 4. An element (1.2) is quasi-invertible in $F$ if and only if

$$\inf_{|m| \geq 1} \left\{ \left\| 1 + \lambda_{nm} |\lambda_{nm}| (|m|!)^{c_2} a_m \right\| \right\} > 0. \quad (3.2)$$

The quasi-inverse of $f$ is the function $g(s) = \sum_{m=1}^{\infty} b_m e^{\lambda_{nm} s}$, where

$$b_m = \frac{-a_m}{1 + |\lambda_{nm}| (|m|!)^{c_2} a_m}. \quad (3.3)$$

Proof. Let $f \in F$ be quasi-invertible, then there exists $g \in F$ such that $f(s) \ast g(s) = 0$. This implies

$$a_m + b_m + \lambda_{nm} |\lambda_{nm}| (|m|!)^{c_2} a_m b_m = 0 \text{ for } |m| \geq 1.$$

Suppose that (3.2) does not hold, that is,

$$\inf_{|m| \geq 1} \left\{ \left\| 1 + \lambda_{nm} |\lambda_{nm}| (|m|!)^{c_2} a_m \right\| \right\} = 0. \quad (3.4)$$

There exists a subsequence $\{m_t\}$ of a sequence of indices $\{m\}$ such that

$$|\lambda_{nm_{m_t}}| (|m_t|!)^{c_2} \|a_{m_t}\| = -1 \text{ as } |t| \to \infty. \quad (3.5)$$

Now (3.3) implies that

$$|\lambda_{nm_{m_t}}| (|m_t|!)^{c_2} \|b_{m_t}\| \geq \frac{|\lambda_{nm_{m_t}}| (|m_t|!)^{c_2} \|a_{m_t}\|}{1 + |\lambda_{nm_{m_t}}| (|m_t|!)^{c_2} a_{m_t}}.$$  

Using (3.4) and (3.5), we see that

$$\|g_t(s)\| \to \infty \text{ as } |t| \to \infty,$$

which is a contradiction.

Conversely, if (3.2) holds, then the function $g$ belongs to $F$ and

$$f(s) \ast g(s) = \sum_{m=1}^{\infty} \{a_m + b_m + \lambda_{nm} |\lambda_{nm}| (|m|!)^{c_2} a_m b_m\} e^{\lambda_{nm} s}$$

$$= 0.$$

Hence $f$ is quasi-invertible. This completes the proof of the theorem. \qed
Definition 2. Let \( f \in F \). The set
\[ \sigma(f) = \{ l \in C : f - le \text{ is not invertible} \} \]
is called the spectrum of \( f \).

Theorem 5. The spectrum \( \sigma(f) \), where \( f \in F \), is precisely of the form
\[ \sigma(f) = \text{cl} \left\{ |\lambda_{nm}|^{c_1}|\lambda_{mn}|\left(|m|!\right)^{c_2}a_m : |m| \geq 1 \right\}. \]

Proof. By Theorem 3, \( f(s) = \sum_{m=1}^\infty a_m e^{\lambda_{nm}s} \) is invertible in \( F \) if and only if the sequence \( (3.1) \) is a bounded. Thus, the function \( \{ f(s) - \alpha e(s) \} \) is not invertible if and only if
\[ \{ \left( |a_m - \alpha e|^{c_1}|\lambda_{nm}|\left(|m|!\right)^{-c_2}\right) \}
\]
is not bounded. This is possible only if there exists a subsequence \( \{ m_r \} \) of the sequence of indicies \( \{ m \} \) such that
\[ \left| |\lambda_{nr}|^{c_1}|\lambda_{mn}|\left(|m_r|!\right)^{c_2}a_{m_r} - \alpha \right| \rightarrow 0 \text{ as } |r| \rightarrow \infty. \]
Thus \( \alpha \in \text{cl} \left\{ |\lambda_{nm}|^{c_1}|\lambda_{mn}|\left(|m|!\right)^{c_2}a_m : |m| \geq 1 \right\} \). The proof is complete. \( \square \)

Theorem 6. Every continuous linear functional \( \theta : F \to E \) is of the form
\[ \theta(f) = \sum_{m=1}^\infty a_m p_m |\lambda_{nm}|^{c_1}|\lambda_{mn}|\left(|m|!\right)^{c_2}, \]
where \( f \) is defined by \( (1.2) \) and \( \{ p_m \} \) is a bounded sequence in \( E \).

Proof. Assume that \( \theta : F \to E \) is a continuous linear functional. Since \( \theta \) is continuous,
\[ \theta(f) = \theta \left( \lim_{M \to \infty} f^{(M)} \right), \]
where
\[ f^{(M)}(s) = \sum_{m=1}^M a_m e^{\lambda_{nm}s}. \]
Define a sequence \( \{ f_m \} \subseteq F \) as
\[ f_m(s) = e |\lambda_{nm}|^{-c_1}|\lambda_{mn}|\left(|m|!\right)^{-c_2}e^{\lambda_{nm}s}. \]
Therefore, using also the linearity of \( \theta \), we have
\[ \theta(f) = \theta \left( \lim_{M \to \infty} \sum_{m=1}^M a_m |\lambda_{nm}|^{c_1}|\lambda_{mn}|\left(|m|!\right)^{c_2}f_m \right). \]
Denoting $\theta(f_m) = p_m$, we may write

$$
\theta(f) = \lim_{M \to \infty} \sum_{m=1}^{M} a_m p_m \left| \lambda_{nm} \right|^{c_1 \left| \lambda_{nm} \right|} \left( |m|! \right)^{c_2} e^{\lambda_{nm}(s+\tau)}.
$$

We now show that $\{p_m\}$ is a bounded sequence in $E$. Indeed,

$$
\|p_m\| = \|\theta(f_m)\| \leq \tau \|f_m\|
$$

and $\|f_m\| = 1$ imply

$$
\|p_m\| \leq \tau.
$$

Thus $\{p_m\}$ is a bounded sequence in $E$. This proves the theorem.

**Theorem 7.** Let $f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{nm}s} \in F$, where $a_m \neq 0$ for every $|m| \geq 1$, and let $K \subset \mathbb{C}^n$ be a set having at least one finite limit point. Define

$$
f_\tau(s) = \sum_{m=1}^{\infty} a_m \left| \lambda_{nm} \right|^{-c_1 \left| \lambda_{nm} \right|} \left( |m|! \right)^{-c_2} e^{\lambda_{nm}(s+\tau)}.
$$

Then the set $A_f = \{f_\tau : \tau \in K\}$ is a total set with respect to the family of continuous linear functionals $\phi : F \to E$.

**Proof.** Note that, for all $\tau \in \mathbb{C}^n$,

$$
\|f_\tau\| = \sum_{m=1}^{\infty} \|a_m\| e^{\Re(\lambda_{nm}\tau)}.
$$

Since (1.2) is an entire Dirichlet series which converges absolutely in the whole complex plane, the series on the right hand side of the above equality is convergent for every $\tau \in K$. Hence $f_\tau(s) \in F$ for every $\tau \in K$.

Let $\phi$ be a continuous linear functional such that $\phi(A_f) \equiv 0$, that is $\phi(f_\tau) = 0$ for all $\tau \in K$. Then, by Theorem 6,

$$
h(s) = \sum_{m=1}^{\infty} a_m p_m e^{\lambda_{nm}s} = 0 \text{ for all } \tau \in K.
$$

Since $\{p_m\}$ is a bounded sequence in $E$ and $f \in F$, the function $h$ also belongs to $F$. But

$$
h(\tau) = \sum_{m=1}^{\infty} a_m p_m e^{\lambda_{nm}\tau} = 0 \text{ for all } \tau \in K.
$$

Since $K$ has a finite limit point, we have that $h \equiv 0$. This however implies that

$$
a_m p_m = 0 \text{ for all } |m| \geq 1,
$$
The assumption $a_m \neq 0$ implies that $p_m = 0$ for all $|m| \geq 1$. Thus $\phi = 0$ and the proof is complete. \hspace{1cm} \Box

References


