ACTA ET COMMENTATIONES UNIVERSITATIS TARTUENSIS DE MATHEMATICA
Volume 21, Number 2, December 2017
Available online at http://acutm.math.ut.ee

Uncertain fuzzy Ostrowski type inequalities for the generalized (s,m)-preinvex Godunova–Levin functions of second kind

Artion Kashuri and Rozana Liko

Abstract. In the present paper, the notion of the generalized (s,m)-preinvex Godunova–Levin function of second kind is introduced and some uncertain fuzzy Ostrowski type inequalities for the generalized (s,m)-preinvex Godunova–Levin functions of second kind via classical integrals and Riemann–Liouville fractional integrals are established.

1. Introduction and preliminaries

The following notation are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. Let $\mathbb{R}^n$ denote an n-dimensional vector space. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ denotes the interior of K. We denote the set of all fuzzy numbers on the set of real numbers $\mathbb{R}$ by $\mathbb{R}_\mathcal{F}$. The space of fuzzy Lebesgue integrable functions on the interval $[a,b]$ is denoted by $L_\mathcal{F}[a,b]$. The space of fuzzy continuous functions on the interval $[a,b]$ is denoted by $C_\mathcal{F}[a,b]$.

The following result is known in the literature as the Ostrowski inequality (see [13]), which gives an upper bound for the approximation of the integral average $(b-a)^{-1}\displaystyle\int_a^b f(t)dt$ by the value $f(x)$ at a point $x \in [a,b]$.

Theorem 1.1. *Let $f : I \longrightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping which is differentiable in the interior I° of I, and let $a, b \in I^\circ$ with $a < b$. If*

Received June 19, 2016.
2010 *Mathematics Subject Classification.* 26A33, 26A51, 33B15, 26B25, 26D07, 26D10, 26D15, 03E72.
Key words and phrases. Euler beta function, Euler gamma function, (s,m)-Godunova–Levin function, fuzzy number, fuzzy Ostrowski inequality, Hölder's inequality, fuzzy fractional Riemann–Liouville operator, m-invex, P-function.
http://dx.doi.org/10.12697/ACUTM.2017.21.15

$|f'(x)| \leq M$ *for all* $x \in [a,b]$, *then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq M(b-a)\left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right], \quad x \in [a,b]. \quad (1.1)$$

In the recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Ostrowski type inequalities (see [13], [21]).

Definition 1.2 (see [6])**.** A nonnegative function $f : I \longrightarrow [0,+\infty)$ is said to be a P-function, or P-convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad x, y \in I,\ t \in [0,1].$$

Definition 1.3 (see [9])**.** A function $f : I \longrightarrow [0,+\infty)$ is said to be a Godunova–Levin function, briefly $f \in Q(I)$, if f is nonnegative and for all $x, y \in I$, $t \in (0,1)$, it holds that

$$f(tx + (1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}.$$

The class $Q(I)$ was firstly described in [9] by Godunova and Levin. Some further properties of it are given in [6], [15], [16]. Among others, let us note that nonnegative monotone and nonnegative convex functions belong to this class of functions.

Definition 1.4 (see [17])**.** A function $f : I \longrightarrow [0,+\infty)$ is said to be an (s,m)-Godunova–Levin function of first kind, briefly $f \in Q^1_{(s,m)}$, if for all $s, m \in (0,1]$, it holds that

$$f(tx + m(1-t)y) \leq \frac{f(x)}{t^s} + \frac{mf(y)}{1-t^s}, \quad x, y \in I,\ t \in (0,1). \quad (1.2)$$

We would like to mention that Definition 1.4 was also introduced and studied by Li et al. [12] independently. For $m = 1$ in Definition 1.4, we have the definition of s-Godunova–Levin functions of first kind, which was introduced and investigated by Noor et al. [18].

Definition 1.5 (see [17])**.** A function $f : I \longrightarrow [0,+\infty)$ is said to be an (s,m)-Godunova–Levin function of second kind, briefly $f \in Q^2_{(s,m)}$, if (1.2) holds for all $s \in [0,1]$, $m \in (0,1]$.

It is obvious that, for $s = 0,\ m = 1$, (s,m)-Godunova–Levin functions of second kind reduce to Definition 1.2 of P-functions. If $s = 1,\ m = 1$, then they reduce to Godunova–Levin functions. For $m = 1$, we have the definition of s-Godunova–Levin function of second kind introduced and studied by Dragomir [5].

Definition 1.6 (see [3])**.** A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true. For more details see [3], [22].

Definition 1.7 (see [19])**.** A function f defined on an invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to η, if for every $x, y \in K$ and $t \in [0,1]$, it holds that

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

Definition 1.8 (see [7])**.** A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\eta(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 1.9. In Definition 1.8, under certain conditions, the mapping $\eta(y,x,m)$ may reduce to $\eta(y,x)$. For example when $m = 1$, then the m-invex set degenerates to an invex set on K.

We next give a new definition of generalized (s,m)-preinvex Godunova–Levin function of second kind.

Definition 1.10. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$. If, for $f : K \longrightarrow \mathbb{R}$ and any fixed $s \in [0,1]$ and $m \in (0,1]$, the inequality

$$f(my + t\eta(x,y,m)) \leq \frac{f(x)}{t^s} + \frac{mf(y)}{(1-t)^s} \tag{1.3}$$

holds for all $x, y \in K$, $t \in (0,1)$, then we say that f is a generalized (s,m)-preinvex Godunova–Levin function of second kind with respect to η, and write $f \in Q^{\star 2}_{(s,m)}$.

Remark 1.11. In Definition 1.10, it is worthwhile to note that a generalized (s,m)-preinvex Godunova–Levin function of second kind is an (s,m)-Godunova–Levin function of second kind on K with respect to $\eta(x,y,m) = x - my$.

Fractional calculus (see [14]) was introduced at the end of the nineteenth century by Liouville and Riemann. It has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 1.12. Let $f \in C_{\mathcal{F}}[a,b]$ and $0 < \alpha \leq 1$. The fuzzy fractional left Riemann–Liouville operator $I^{\alpha}_{a+}f$ and fuzzy fractional right Riemann–Liouville operator $I^{\alpha}_{b-}f$ are defined, respectively, by

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x \in [a,b],$$

and

$$I^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x \in [a,b],$$

where $\Gamma(\alpha)$ is the Euler gamma function. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended the study of fractional Hermite–Hadamard, Grüss, or Ostrowski type inequalities to functions of different classes (see [14]).

Since fuzziness is a natural reality different than randomness and determinism, Anastassiou (see [1],[2]) established fuzzy Ostrowski's inequalities. These inequalities have been applied to Euler's beta mapping (see [20]) and special means such as the arithmetic mean, geometric mean, harmonic mean, and others. Concepts of fuzzy Riemann integrals were introduced by Wu [4]. A fuzzy Riemann integral is a closed interval whose end points are the classical Riemann integrals.

Definition 1.13 (see [4])**.** A fuzzy number is a mapping $u : \mathbb{R} \longrightarrow [0,1]$ with the following properties:

(1) u is upper semi-continuous, i.e., for any small positive $\epsilon, u(x)$ always is less than $u(x_0) + \epsilon$ for all x in some neighborhood of x_0;
(2) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$;
(3) u is normal, i.e., there exists $x_0 \in \mathbb{R}$ for which $u(x_0) = 1$;
(4) the closure of the support of u is compact, where supp $u = \big\{x \in \mathbb{R} |\, u(x) > 0\big\}$.

Definition 1.14 (see [8])**.** An arbitrary fuzzy number is represented by an ordered pair of functions $(u^-(\alpha), u^+(\alpha))$, $0 \leq \alpha \leq 1$, that satisfies the following requirements:

(1) $u^-(\alpha)$ is a bounded left continuous nondecreasing function over $[0,1]$, with respect to any α;
(2) $u^+(\alpha)$ is a bounded left continuous nonincreasing function over $[0,1]$, with respect to any α;
(3) $u^-(\alpha) \leq u^+(\alpha)$, $0 \leq \alpha \leq 1$.

The α-level set $[u]^{\alpha}$ of a fuzzy set u on $\mathbb{R}$ is defined as

$$[u]^{\alpha} = \big\{x \in \mathbb{R} |\, u(x) \geq \alpha\big\}, \text{ for each } \alpha \in (0,1], \tag{1.4}$$

and as

$$[u]^0 = \overline{\bigcup_{\alpha\in(0,1]} [u]^\alpha}, \quad \text{for } \alpha = 0, \tag{1.5}$$

where $\overline{A}$ denotes the closure of A.

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define the sum $[u \oplus v]^\alpha$ and the product $[\lambda \odot u]^\alpha$ by

$$[u \oplus v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [\lambda \odot u]^\alpha = \lambda[u]^\alpha, \quad \alpha \in [0,1],$$

where $[u]^\alpha + [v]^\alpha$ means the usual addition of two integrals (as subsets of $\mathbb{R}$) and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow [0, +\infty)$ by

$$D(u,v) := \sup_{\alpha\in[0,1]} \max\left\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\right\}, \tag{1.6}$$

where

$$[v]^\alpha = [v_\alpha^-, v_\alpha^+], \ v \in \mathbb{R}_{\mathcal{F}}. \tag{1.7}$$

It is easy to show that D is a metric on $\mathbb{R}_{\mathcal{F}}$ and $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space with the following properties:

(1) $D(u \oplus w, v \oplus w) = D(u,v), \quad u, v, w \in \mathbb{R}_{\mathcal{F}}$;
(2) $D(k \odot u, k \odot v) = |k| D(u,v), \quad u, v \in \mathbb{R}_{\mathcal{F}}, \ \forall k \in \mathbb{R}$;
(3) $D(u \oplus v, w \oplus e) = D(u,w) + D(v,e), \quad u, v, w, e \in \mathbb{R}_{\mathcal{F}}$;
(4) $D(u \oplus v, \tilde{0}) \le D(u, \tilde{0}) + D(v, \tilde{0}), \quad u, v \in \mathbb{R}_{\mathcal{F}}$;
(5) $D(u \oplus v, w) \le D(u,w) + D(v, \tilde{0}), \quad u, v, w \in \mathbb{R}_{\mathcal{F}}$.

where $\tilde{0} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\tilde{0}(x) = 0$ for all $x \in \mathbb{R}$.

Let $u, v \in \mathbb{R}_{\mathcal{F}}$. If there exists a $w \in \mathbb{R}_{\mathcal{F}}$ such that $u = v \oplus w$, then we call w the H-difference of u and v; it is denoted by $w = u \ominus v$.

Let $[a,b] \subset \mathbb{R}$, we say that the function $f : [a,b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ is fuzzy Riemann integrable on $[a,b]$ if there exist $I \in \mathbb{R}_{\mathcal{F}}$ satisfying the following property: for every $\epsilon > 0$ there exists $\delta > 0$ such that for any partition $P = \{a = x_0 < x_1 < \ldots < x_n\}$ of $[a,b]$ with $\|P\| < \delta$, and any points $\xi_i \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$, we have

$$D\left(\sum_{i=0}^{n-1} (x_{i+1} - x_i) \odot f(\xi_i), \ I\right) < \epsilon.$$

We write

$$I := (FR) \int_a^b f(x)dx. \tag{1.8}$$

We also call an f as above (FR)-integrable.

Let $f : [a, b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then it is easy to show that $(FR)\displaystyle\int_a^b f(x)dx$ exists and it belongs to $\mathbb{R}_{\mathcal{F}}$. Furthermore,

$$\left[\int_a^b f(x)dx\right]^{\alpha} = \left[\int_a^b f_{\alpha}^{-}(x)dx, \int_a^b f_{\alpha}^{+}(x)dx\right]^{\alpha}, \quad \alpha \in [0, 1].$$

Let $f, g \in C_{\mathcal{F}}[a, b]$ and $c_1, c_2 \in [a, b]$. Then

$$(FR)\int_a^b \Big(c_1 f(x) + c_2 g(x)\Big)dx = c_1(FR)\int_a^b f(x)dx + c_2(FR)\int_a^b g(x)dx.$$

Also, if $f, g : [a, b] \longrightarrow \mathbb{R}_{\mathcal{F}}$ are fuzzy continuous functions, then the function $F : [a, b] \longrightarrow [0, +\infty)$, defined by $F(x) = D\Big(f(x), g(x)\Big)$, is continuous on $[a, b]$ and

$$D\left((FR)\int_a^b f(x)dx, (FR)\int_a^b g(x)dx\right) \leq \int_a^b D\Big(f(x), g(x)\Big)dx. \qquad (1.9)$$

In the following we present the definition of H-differentiability which will be used in the remaining part of the paper.

Definition 1.15. Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, $\beta > 0$. A function $f : T \longrightarrow \mathbb{R}_{\mathcal{F}}$ is H-differentiable at $x \in T$ if there exists $f'(x) \in \mathbb{R}_{\mathcal{F}}$ such that, in the metric D,

(1) $\displaystyle\lim_{h \longrightarrow 0^+} \frac{f(x+h) \ominus f(x)}{h} = \lim_{h \longrightarrow 0^+} \frac{f(x) \ominus f(x-h)}{h} = f'(x)$, or

(2) $\displaystyle\lim_{h \longrightarrow 0^+} \frac{f(x) \ominus f(x+h)}{-h} = \lim_{h \longrightarrow 0^+} \frac{f(x-h) \ominus f(x)}{-h} = f'(x)$, or

(3) $\displaystyle\lim_{h \longrightarrow 0^+} \frac{f(x+h) \ominus f(x)}{h} = \lim_{h \longrightarrow 0^+} \frac{f(x-h) \ominus f(x)}{-h} = f'(x)$, or

(4) $\displaystyle\lim_{h \longrightarrow 0^+} \frac{f(x) \ominus f(x+h)}{-h} = \lim_{h \longrightarrow 0^+} \frac{f(x) \ominus f(x-h)}{h} = f'(x)$.

We call f' the derivative, or H-derivative, of f at x. If f is H-differentiable at any $x \in T$, we call the f differentiable, or H-differentiable.

In the present paper, the notion of (s, m)-preinvex Godunova–Levin function of second kind is applied to establish uncertain fuzzy Ostrowski type integral inequalties. The paper is organized as follows. In Section 2, some uncertain fuzzy Ostrowski type inequalities for generalized (s, m)-preinvex Godunova–Levin functions of second kind via fractional integrals are given. Also, some special cases of our theorems via classical integrals are deduced. In Section 3, some conclusions and an insight to the future research are given.

2. Uncertain fuzzy Ostrowski type inequalities for generalized (s, m)-preinvex Godunova–Levin functions of second kind via fractional integrals

In this section, in order to prove our main results regarding some uncertain fuzzy Ostrowski type inequalities for generalized (s, m)-preinvex Godunova–Levin functions of second kind via fractional integrals, we need the following lemma.

Lemma 2.1. *Let $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$, and let $a, b \in K$, $a < b$, satisfy $ma < ma + \eta(b,a,m)$. Assume that $f : K \longrightarrow \mathbb{R}_{\mathcal{F}}$ is a differentiable function on K° and $f' \in C_{\mathcal{F}}[ma, ma+\eta(b,a,m)] \bigcap L_{\mathcal{F}}[ma, ma+\eta(b,a,m)]$. Then, for each $x \in [ma, ma+\eta(b,a,m)]$ and $0 < \alpha \leq 1$, we have that $G = H \oplus J$, where*

$$G = \frac{\eta^{\alpha}(x,a,m) \odot f(ma+\eta(x,a,m)) \ominus \eta^{\alpha}(x,b,m) \odot f(mb+\eta(x,b,m))}{\eta(b,a,m)},$$

$$H = \frac{\Gamma(\alpha+1)}{\eta(b,a,m)} \odot \left[I^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) \ominus I^{\alpha}_{(mb+\eta(x,b,m))-} f(mb) \right],$$

$$\begin{aligned} J = &\frac{\eta^{\alpha+1}(x,a,m)}{\eta(b,a,m)} \odot (FR) \int_0^1 t^{\alpha} \odot f'(ma+t\eta(x,a,m))dt \\ &\ominus \frac{\eta^{\alpha+1}(x,b,m)}{\eta(b,a,m)} \odot (FR) \int_0^1 t^{\alpha} \odot f'(mb+t\eta(x,b,m))dt. \end{aligned}$$

Here $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$ is the Euler gamma function.

Proof. By integration by parts and using properties of the α-cut of fuzzy numbers, we have the following identities:

$$\begin{aligned} J = &\frac{\eta^{\alpha+1}(x,a,m)}{\eta(b,a,m)} \odot \left[t^{\alpha} \odot \frac{f(ma+t\eta(x,a,m))}{\eta(x,a,m)} \Big|_0^1 \right. \\ &\left. \ominus \alpha \odot (FR) \int_0^1 \frac{t^{\alpha-1} \odot f(ma+t\eta(x,a,m))}{\eta(x,a,m)} dt \right] \\ &\ominus \frac{\eta^{\alpha+1}(x,b,m)}{\eta(b,a,m)} \odot \left[t^{\alpha} \odot \frac{f(mb+t\eta(x,b,m))}{\eta(x,b,m)} \Big|_0^1 \right. \\ &\left. \ominus \alpha \odot (FR) \int_0^1 \frac{t^{\alpha-1} \odot f(mb+t\eta(x,b,m))}{\eta(x,b,m)} dt \right] \\ = &\frac{\eta^{\alpha+1}(x,a,m)}{\eta(b,a,m)} \odot \left[\frac{f(ma+\eta(x,a,m))}{\eta(x,a,m)} \right. \\ &\left. \ominus \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(x,a,m)} \odot I^{\alpha}_{(ma+\eta(x,a,m))-} f(ma) \right] \end{aligned}$$

$$\ominus \frac{\eta^{\alpha+1}(x,b,m)}{\eta(b,a,m)} \odot \left[\frac{f(mb+\eta(x,b,m))}{\eta(x,b,m)}\right.$$
$$\left.\ominus \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(x,b,m)} \odot I^{\alpha}_{(mb+\eta(x,b,m))-} f(mb)\right]$$
$$= G \ominus H.$$

□

By using Lemma 2.1, one can obtain the following results.

Theorem 2.2. *Let $A \subseteq [0,+\infty)$ be an open m-invex subset with respect to $\eta : A \times A \times (0,1] \longrightarrow [0,+\infty)$ for any fixed $m \in (0,1]$, and let $a, b \in A$, $a < b$, satisfy $ma < ma + \eta(b,a,m)$. Assume that $f : A \longrightarrow \mathbb{R}_{\mathcal{F}}$ is a differentiable function on A°, with $f' \in C_{\mathcal{F}}[ma, ma+\eta(b,a,m)] \bigcap L_{\mathcal{F}}[ma, ma+\eta(b,a,m)]$. If $D\left(f'(x), \tilde{0}\right)$ is a generalized (s,m)-preinvex Godunova–Levin function of second kind on $[ma, ma+\eta(b,a,m)]$ for any fixed $s \in [0,1)$, then for each $x \in [ma, ma+\eta(b,a,m)]$ and $0 < \alpha \leq 1$, we have*

$$\begin{aligned} D(G,H) \leq\ & \frac{1}{(\alpha-s+1)|\eta(b,a,m)|} D(f'(x),\tilde{0}) \\ & \times \left[|\eta(x,a,m)|^{\alpha+1} + |\eta(x,b,m)|^{\alpha+1}\right] + \frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)|\eta(b,a,m)|} \\ & \times \left[|\eta(x,a,m)|^{\alpha+1} D(f'(a),\tilde{0}) + |\eta(x,b,m)|^{\alpha+1} D(f'(b),\tilde{0})\right]. \end{aligned} \tag{2.1}$$

Proof. Using Lemma 2.1, we have

$$\begin{aligned} D(G,H) \leq\ & \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} D(f'(ma+t\eta(x,a,m)),\tilde{0})dt \\ & + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} D(f'(mb+t\eta(x,b,m)),\tilde{0})dt \\ \leq\ & \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} \left[\frac{1}{t^s} D(f'(x),\tilde{0}) + \frac{m}{(1-t)^s} D(f'(a),\tilde{0})\right] dt \\ & + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \int_0^1 t^{\alpha} \left[\frac{1}{t^s} D(f'(x),\tilde{0}) + \frac{m}{(1-t)^s} D(f'(b),\tilde{0})\right] dt \\ =\ & \frac{1}{(\alpha-s+1)|\eta(b,a,m)|} D(f'(x),\tilde{0}) \left[|\eta(x,a,m)|^{\alpha+1}\right. \\ & \left. + |\eta(x,b,m)|^{\alpha+1}\right] + \frac{m\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)|\eta(b,a,m)|} \\ & \times \left[|\eta(x,a,m)|^{\alpha+1} D(f'(a),\tilde{0}) + |\eta(x,b,m)|^{\alpha+1} D(f'(b),\tilde{0})\right]. \end{aligned} \tag{2.2}$$

□

Corollary 2.3. *Under the assumptions of Theorem 2.2, putting* $\alpha = 1$, *we get*

$$D(P,Q) \leq \frac{1}{(2-s)|\eta(b,a,m)|} D(f'(x),\tilde{0}) \left[\eta^2(x,a,m) + \eta^2(x,b,m)\right] + \frac{m\beta(2,1-s)}{|\eta(b,a,m)|} \left[\eta^2(x,a,m)D(f'(a),\tilde{0}) + \eta^2(x,b,m)D(f'(b),\tilde{0})\right],$$

where

$$P = \frac{\eta(x,a,m) \odot f(ma+\eta(x,a,m)) \ominus \eta(x,b,m) \odot f(mb+\eta(x,b,m))}{\eta(b,a,m)},$$

$$Q = \frac{1}{\eta(b,a,m)} \odot \left[(FR)\int_{ma}^{ma+\eta(x,a,m)} f(u)du \ominus (FR)\int_{mb}^{mb+\eta(x,b,m)} f(u)du\right].$$

Remark 2.4. In Theorem 2.2, if we choose $m = 1$ and $\eta(x,y,m) = x - my$, then inequality (2.1) reduces to

$$D\left(\frac{(x-a)^\alpha \ominus (x-b)^\alpha}{b-a} \odot f(x), \frac{\Gamma(\alpha+1)}{b-a} \odot \left[I^\alpha_{x-}f(a) \ominus I^\alpha_{x-}f(b)\right]\right)$$
$$\leq \frac{1}{(\alpha-s+1)(b-a)} D(f'(x),\tilde{0}) \left[(x-a)^{\alpha+1} + (b-x)^{\alpha+1}\right] + \frac{\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)(b-a)} \left[(x-a)^{\alpha+1}D(f'(a),\tilde{0}) + (b-x)^{\alpha+1}D(f'(b),\tilde{0})\right].$$

The corresponding version for powers of the first derivative is contained in the following results.

Theorem 2.5. *Let* A, a, b, *and* f *be the same as in Theorem 2.2. If* $D^q\left(f'(x),\tilde{0}\right)$ *is a generalized* (s,m)*-preinvex Godunova–Levin function of second kind on* $[ma, ma+\eta(b,a,m)]$ *for any fixed* $s \in [0,1)$, $q > 1$, $p^{-1} + q^{-1} = 1$, *and* $0 < \alpha \leq 1$, *then for each* $x \in [ma, ma+\eta(b,a,m)]$, *we have*

$$D(G,H) \leq \frac{1}{(1+p\alpha)^{1/p}} \frac{1}{(1-s)^{1/q}} \frac{1}{|\eta(b,a,m)|} \times \left\{|\eta(x,a,m)|^{\alpha+1}\left[D^q(f'(x),\tilde{0}) + mD^q(f'(a),\tilde{0})\right]^{\frac{1}{q}} + |\eta(x,b,m)|^{\alpha+1}\left[D^q(f'(x),\tilde{0}) + mD^q(f'(b),\tilde{0})\right]^{\frac{1}{q}}\right\}. \tag{2.3}$$

Proof. Suppose that $q > 1$. Using Lemma 2.1 and Hölder's inequality, in view of the first inequality from (2.2), we have

$$D(G,H) \leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1 D^q(f'(ma+t\eta(x,a,m)),\tilde{0})dt\right)^{\frac{1}{q}} + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 D^q(f'(mb+t\eta(x,b,m)),\tilde{0})dt\right)^{\frac{1}{q}}$$

$$\leq \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}$$

$$\times \left[\int_0^1 \left(\frac{1}{t^s}D^q(f'(x),\tilde{0})+\frac{m}{(1-t)^s}D^q(f'(a),\tilde{0})\right)dt\right]^{\frac{1}{q}}$$

$$+\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}$$

$$\times \left[\int_0^1 \left(\frac{1}{t^s}D^q(f'(x),\tilde{0})+\frac{m}{(1-t)^s}D^q(f'(b),\tilde{0})\right)dt\right]^{\frac{1}{q}}$$

$$=\frac{1}{(1+p\alpha)^{1/p}}\frac{1}{(1-s)^{1/q}}\frac{1}{|\eta(b,a,m)|}$$

$$\times \left\{|\eta(x,a,m)|^{\alpha+1}\left[D^q(f'(x),\tilde{0})+mD^q(f'(a),\tilde{0})\right]^{\frac{1}{q}}\right.$$

$$\left.+|\eta(x,b,m)|^{\alpha+1}\left[D^q(f'(x),\tilde{0})+mD^q(f'(b),\tilde{0})\right]^{\frac{1}{q}}\right\}.$$

□

Corollary 2.6. *Under the assumptions of Theorem 2.5, for $\alpha = 1$, we get*

$$D(P,Q) \leq \frac{1}{(p+1)^{1/p}}\frac{1}{(1-s)^{1/q}}\frac{1}{|\eta(b,a,m)|}$$

$$\times \left\{\eta^2(x,a,m)\left[D^q\left(f'(x),\tilde{0}\right)+mD^q\left(f'(a),\tilde{0}\right)\right]^{\frac{1}{q}}\right.$$

$$\left.+\eta^2(x,b,m)\left[D^q\left(f'(x),\tilde{0}\right)+mD^q\left(f'(b),\tilde{0}\right)\right]^{\frac{1}{q}}\right\}.$$

Remark 2.7. In Theorem 2.5, if we choose $m = 1$ and $\eta(x,y,m) = x-my$, then inequality (2.3) reduces to

$$D\left(\frac{(x-a)^\alpha \ominus (x-b)^\alpha}{b-a}\odot f(x), \frac{\Gamma(\alpha+1)}{b-a}\odot\left[I^\alpha_{x-}f(a)\ominus I^\alpha_{x-}f(b)\right]\right)$$

$$\leq \frac{1}{(1+p\alpha)^{1/p}}\frac{1}{(1-s)^{1/q}}\frac{1}{b-a}$$

$$\times\left\{(x-a)^{\alpha+1}\left[D^q(f'(x),\tilde{0})+D^q(f'(a),\tilde{0})\right]^{\frac{1}{q}}\right.$$

$$\left.+(b-x)^{\alpha+1}\left[D^q(f'(x),\tilde{0})+D^q(f'(b),\tilde{0})\right]^{\frac{1}{q}}\right\}.$$

Theorem 2.8. *Let A, a, b, and f be the same as in Theorem 2.2. If $D^q\left(f'(x),\tilde{0}\right)$ is a generalized (s,m)-preinvex Godunova–Levin function of* *second kind on* $[ma, ma + \eta(b, a, m)]$ *for any fixed* $s \in [0, 1)$, $q \geq 1$ *and* $0 < \alpha \leq 1$, *then for each* $x \in [ma, ma + \eta(b, a, m)]$ *we have*

$$\begin{aligned} D(G,H) \leq & \frac{1}{(1+\alpha)^{1-\frac{1}{q}}} \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \\ & \times \left[\frac{D^q(f'(x),\tilde{0})}{\alpha - s + 1} + mD^q(f'(a),\tilde{0}) \frac{\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha - s + 2)} \right]^{\frac{1}{q}} \\ & + \frac{1}{(1+\alpha)^{1-\frac{1}{q}}} \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \\ & \times \left[\frac{D^q(f'(x),\tilde{0})}{\alpha - s + 1} + mD^q(f'(b),\tilde{0}) \frac{\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha - s + 2)} \right]^{\frac{1}{q}}. \end{aligned} \tag{2.4}$$

Proof. Let $q \geq 1$. Using Lemma 2.1, the first inequality from (2.2), and the well-known power mean inequality, we have

$$\begin{aligned} D(G,H) \leq & \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \left(\int_0^1 t^\alpha dt \right)^{1-\frac{1}{q}} \\ & \times \left(\int_0^1 t^\alpha D^q(f'(ma + t\eta(x,a,m)),\tilde{0}) dt \right)^{\frac{1}{q}} \\ & + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \left(\int_0^1 t^\alpha dt \right)^{1-\frac{1}{q}} \\ & \times \left(\int_0^1 t^\alpha D^q(f'(mb + t\eta(x,b,m)),\tilde{0}) dt \right)^{\frac{1}{q}} \\ \leq & \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \left(\int_0^1 t^\alpha dt \right)^{1-\frac{1}{q}} \\ & \times \left[\int_0^1 t^\alpha \left(\frac{1}{t^s} D^q(f'(x),\tilde{0}) + \frac{m}{(1-t)^s} D^q(f'(a),\tilde{0}) \right) dt \right]^{\frac{1}{q}} \\ & + \frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|} \left(\int_0^1 t^\alpha dt \right)^{1-\frac{1}{q}} \\ & \times \left[\int_0^1 t^\alpha \left(\frac{1}{t^s} D^q(f'(x),\tilde{0}) + \frac{m}{(1-t)^s} D^q(f'(b),\tilde{0}) \right) dt \right]^{\frac{1}{q}} \\ = & \frac{1}{(1+\alpha)^{1-\frac{1}{q}}} \frac{|\eta(x,a,m)|^{\alpha+1}}{|\eta(b,a,m)|} \\ & \times \left[\frac{D^q(f'(x),\tilde{0})}{\alpha - s + 1} + mD^q(f'(a),\tilde{0}) \frac{\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha - s + 2)} \right]^{\frac{1}{q}} \end{aligned}$$

$$+\frac{1}{(1+\alpha)^{1-\frac{1}{q}}}\frac{|\eta(x,b,m)|^{\alpha+1}}{|\eta(b,a,m)|}$$
$$\times\left[\frac{D^q(f'(x),\tilde{0})}{\alpha-s+1}+mD^q(f'(b),\tilde{0})\frac{\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]^{\frac{1}{q}}.$$

□

Corollary 2.9. *Under the assumptions of Theorem 2.8, if we put $\alpha = 1$, we get*

$$D(P,Q)\leq\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{\eta^2(x,a,m)}{|\eta(b,a,m)|}\left[\frac{D^q(f'(x),\tilde{0})}{2-s}+m\beta(2,1-s)D^q(f'(a),\tilde{0})\right]^{\frac{1}{q}}$$
$$+\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\frac{\eta^2(x,b,m)}{|\eta(b,a,m)|}\left[\frac{D^q(f'(x),\tilde{0})}{2-s}+m\beta(2,1-s)D^q(f'(b),\tilde{0})\right]^{\frac{1}{q}}.$$

Remark 2.10. In Theorem 2.8, if we choose $m = 1$ and $\eta(x, y, m) = x-my$, then inequality (2.4) reduces to

$$D\left(\frac{(x-a)^\alpha\ominus(x-b)^\alpha}{b-a}\odot f(x),\frac{\Gamma(\alpha+1)}{b-a}\odot\left[I^\alpha_{x-}f(a)\ominus I^\alpha_{x-}f(b)\right]\right)$$
$$\leq\frac{1}{(1+\alpha)^{1-\frac{1}{q}}}\frac{(x-a)^{\alpha+1}}{b-a}\left[\frac{D^q(f'(x),\tilde{0})}{\alpha-s+1}+D^q(f'(a),\tilde{0})\frac{\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]^{\frac{1}{q}}$$
$$+\frac{1}{(1+\alpha)^{1-\frac{1}{q}}}\frac{(b-x)^{\alpha+1}}{b-a}\left[\frac{D^q(f'(x),\tilde{0})}{\alpha-s+1}+D^q(f'(b),\tilde{0})\frac{\Gamma(\alpha+1)\Gamma(1-s)}{\Gamma(\alpha-s+2)}\right]^{\frac{1}{q}}.$$

Remark 2.11. For $M > 0$ and $q \geq 1$, if $D(f'(x),\tilde{0}) \leq M$ or $D^q(f'(x),\tilde{0}) \leq M$, then by our theorems of this paper, we can get some special kinds of fuzzy Ostrowski type inequalities.

3. Conclusions

In this paper, we investigated uncertain fuzzy Ostrowski type inequalities for the functions whose derivatives are generalized (s, m)-preinvex Godunova–Levin functions of second kind via Riemann–Liouville fractional integrals. Some special cases of our theorems via classical integrals are also deduced. These results can be applied to find new inequalities for special means such as geometric, arithmetic, logarithmic means etc.

We hope that this new class of generalized (s, m)-preinvex Godunova–Levin functions of second kinds could motivate for fellow researchers and scientists working in the same domain.

We conclude that our methods considered here may be useful for further investigations concerning uncertain fuzzy and fuzzy Ostrowski and Hermite–Hadamard type integral inequalities for various kinds of preinvex functions involving classical integrals, Riemann–Liouville fractional integrals, k-fractional integrals, local fractional integrals, fractional integral operators, q-calculus, (p, q)-calculus, time scale calculus, and conformable fractional integrals.

Acknowledgements. We thank anonymous referees and editors for their valuable suggestions regarding the manuscript.

References

[1] G. A. Anastassiou, *Fuzzy Ostrowski type inequalities*, Comput. Appl. Math. **22** (2003), 279–292.

[2] G. A. Anastassiou, *Fuzzy Mathematics; Approximation Theory*, Studies in Fuzziness and Soft Computing **251**, Springer-Verlag, Berlin, 2010.

[3] T. Antczak, *Mean value in invexity analysis*, Nonlinear Anal. **60** (2005), 1473–1484.

[4] W. Congxin and M. Ming, *On embedding problem of fuzzy number space: Part 1*, Fuzzy Sets and Systems **44** (1991), 33–38.

[5] S. S. Dragomir, *Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces*, Proyecciones **34**(4) (2015), 323–341.

[6] S. S. Dragomir, J. Pečarić, and L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995), 335–341.

[7] T. S. Du, J. G. Liao, and Y. J. Li, *Properties and integral inequalities of Hadamard-Simpson type for the generalized (s, m)-preinvex functions*, J. Nonlinear Sci. Appl. **9** (2016), 3112–3126.

[8] D. Dubois and H. Prade, *Fundamentals of Fuzzy Sets*, Kluwer Academic Publishers, Boston, 2000.

[9] E. K. Godunova and V. I. Levin, *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, in: Numerical Mathematics and Mathematical Physics, Moscov. Gos. Ped. Inst., Moskov, 1985, pp. 138–142, 166. (Russian)

[10] A. Kashuri and R. Liko, *Ostrowski type fractional integral inequalities for generalized (s, m, φ)-preinvex functions*, Aust. J. Math. Anal. Appl. **13** (1) (2016), Article 16, 1–11.

[11] A. Kashuri and R. Liko, *Generalizations of Hermite-Hadamard and Ostrowski type inequalities for MT_m-preinvex functions*, Proyecciones **36**(1) (2017), 45–80.

[12] M. Li, J. R. Wang, and W. Wei, *Some fractional Hermite-Hadamard inequalities for convex and Godunova-Levin functions*, Facta Univ. Ser. Math. Inform. **30**(2) (2015), 195–208.

[13] W. Liu, W. Wen, and J. Park, *Ostrowski type fractional integral inequalities for MT-convex functions*, Miskolc Math. Notes **16**(1) (2015), 249–256.

[14] W. Liu, W. Wen, and J. Park, *Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl. **9** (2016), 766–777.

[15] D. S. Mitrinović and J. Pečarić, *Note on a class of functions of Godunova and Levin*, C. R. Math. Rep. Acad. Sci. Can. **12** (1990), 33–36.

[16] D. S. Mitrinović, J. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Mathematics and its Applications (East European Series) **61**, Kluwer Academic Publishers, Dordrecht, 1993.

[17] M. A. Noor, K. I. Noor, and M. U. Awan, *Fractional Ostrowski inequalities for (s, m)-Godunova-Levin functions*, Facta Univ. Ser. Math. Inform. **30**(4), (2015), 489–499.

[18] M. A. Noor, K. I. Noor, M. U. Awan, and S. Khan, *Fractional Hermite-Hadamard inequalities for some new classes of Godunova-Levin functions*, Appl. Math. Inf. Sci. **8**(6) (2014), 2865–2872.

[19] R. Pini, *Invexity and generalized convexity*, Optimization **22** (1991), 513–525.

[20] K. L. Tseng, *Improvement of some inequalities of Ostrowski type and their application*, Taiwanese J. Math. **12** (2008), 2427–2441.

[21] M. Tunç, *Ostrowski type inequalities for functions whose derivatives are MT-convex*, J. Comput. Anal. Appl. **17**(4) (2014), 691–696.

[22] X. M. Yang, X. Q. Yang, and K. L. Teo, *Generalized invexity and generalized invariant monotonicity*, J. Optim. Theory Appl. **117** (2003) 607–625.

Department of Mathematics, Faculty of Technical Science, University "Ismail Qemali", Vlora, Albania

E-mail address: artionkashuri@gmail.com

E-mail address: rozanaliko86@gmail.com