

A note on a new unique range set with truncated multiplicity

ABHIJIT BANERJEE AND SANTANU DHAR

ABSTRACT. We introduce a new polynomial whose zero set forms a unique range set for meromorphic function with 11 elements under relaxed sharing hypothesis.

1. Introduction and definitions

Throughout the paper, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By a meromorphic function we shall always mean a meromorphic function in the complex plane \mathbb{C} . We adopt the usual notation of Nevanlinna theory as explained in [10]. By E and I we denote any set of finite and infinite linear measure, respectively. For any non-constant meromorphic function $h(z)$, we define $S(r, h)$ by $S(r, h) = o(T(r, h))$ where $r \rightarrow \infty$, $r \notin E$.

Let f be a non-constant meromorphic function, let $a \in \overline{\mathbb{C}}$, and let p be a positive integer. We denote by $E(a, f)$ the set of zeros of $f(z) - a$ (counting multiplicity) and by $E_p(a, f)$ the set of zeros of $f(z) - a$ with multiplicity $\leq p$ (counting multiplicity).

Let $S \subset \mathbb{C}$. Set

$$E(S, f) = \bigcup_{a \in S} E(a, f), \quad E_p(S, f) = \bigcup_{a \in S} E_p(a, f).$$

Then for two non-constant meromorphic functions f and g we say that f, g share the set S truncated p if $E_p(S, f) = E_p(S, g)$. Obviously the condition $E_p(S, f) = E_p(S, g)$ implies $E_j(S, f) = E_j(S, g)$ for all $1 \leq j \leq p$.

The inception of set sharing problem in the realm of the theory of meromorphic function was due to the famous ‘‘Gross Question’’ (see [8]) which is as follows.

Received October 27, 2015.

2010 *Mathematics Subject Classification.* 30D35.

Key words and phrases. Meromorphic function, unique range set, shared set, truncated multiplicity.

<http://dx.doi.org/10.12697/ACUTM.2017.21.13>

Question A. Can one find two (or possibly even one) finite sets S_j ($j = 1, 2$) such that any two non-constant entire functions f and g satisfying $E(S_j, f) = E(S_j, g)$ for $j = 1, 2$ must be identical?

Gradually the research to find the possible answer of *Question A* corresponding to meromorphic functions has become one of the most prominent branches of the uniqueness theory. Later on many analogous questions were raised by many researchers pertinent to their investigations. It is needless to say that investigations for possible answers of these questions have enriched the uniqueness theory vis-à-vis the value distribution theory. Meanwhile, Gross and Yang [9] (see also [13]) introduced the new idea of a unique range set for meromorphic functions (URSM, in brief) in the following manner.

Definition 1.1 (see [9]). A set S such that for any two non-constant entire (meromorphic) functions f and g the condition $E(S, f) = E(S, g)$ implies $f \equiv g$ is called a unique range set of entire (meromorphic) functions. We call it URSE (URSM) in short.

Recently the definition of unique range sets have been generalized in [4] as follows.

Definition 1.2 (see [4]). A set S is called a $URSM_p$ ($URSE_p$) if for any two non-constant meromorphic (entire) functions f and g the equality $E_p(S, f) = E_p(S, g)$ implies $f \equiv g$.

Relevant to definition 1.1, Yi [17] introduced in 1996 a URSM with 13 elements. Two years later Frank and Reinders [6] introduced another URSM with 11 elements. Till date we have another two URSM's, one by Banerjee [3] and the other one by Alzahary [1], and both of these sets are of cardinality 11. In this paper we introduce another $URSM_3$ with 11 elements. Throughout the paper we shall denote by $P(z)$ the following polynomial:

$$P(z) = z^n + 2n(n-2m)z^{n-m} + n(n-2m)(n-m)^2z^{n-2m} + c, \quad (1.1)$$

where $n, m \in \mathbb{N}$, $\gcd(m, n) = 1$, $c \in \mathbb{C}$ are such that $P(z)$ has no multiple zero, and

$$\beta_i = - (c_i^n + 2n(n-2m)c_i^{n-m} + n(n-2m)(n-m)^2c_i^{n-2m}),$$

where c_i are the roots of the equation $z^m + (n-m)(n-2m) = 0$ for $i = 1, 2, \dots, m$.

The following theorem is the main result of this paper.

Theorem 1.1. *Let $S = \{z : P(z) = 0\}$. If one of the conditions*

- (i) $p \geq 3$ for $n \geq 2m + 9$ ($n \geq 2m + 5$),
- (ii) $p = 2$ for $n \geq 2m + 10$ ($n \geq 2m + 5$),
- (iii) $p = 1$ for $n \geq 2m + 13$ ($n \geq 2m + 7$)

holds, then S is a $URSM_p$ ($URSE_p$).

We have already mentioned that readers are referred to go through [10] for standard notation and definitions of the value distribution theory but below we explain some notation which is frequently used in the paper.

Definition 1.3 (see [11]). For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $N(r, a; f | = 1)$ the counting function of simple a -points of f . For a positive integer m , we denote by $N(r, a; f | \leq p)$ ($N(r, a; f | \geq p)$) the counting function of those a -points of f whose multiplicities are not greater (less) than p , where each a -point is counted according to its multiplicity.

$\overline{N}(r, a; f | \leq p)$ ($\overline{N}(r, a; f | \geq p)$) are defined similarly, where in counting the a -points of f we ignore the multiplicities.

Also $N(r, a; f | < p)$, $N(r, a; f | > p)$, $\overline{N}(r, a; f | < p)$, and $\overline{N}(r, a; f | > p)$ are defined analogously.

Definition 1.4 (see [5]). Let f and g be two non-constant meromorphic functions and let p be a positive integer such that $E_p(a; f) = E_p(a; g)$, where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a -point of f with multiplicity $s > 0$, or an a -point of g with multiplicity $q > 0$. We denote by $\overline{N}_L^{(p)}(r, a; f)$ ($\overline{N}_L^{(p)}(r, a; g)$) the counting function of those a -points of f and g , where $s > q$ ($q > s$) and each a -point is counted only once.

Definition 1.5 (see [5]). Let p be a positive integer and, for $a \in \mathbb{C}$, let $E_p(a; f) = E_p(a; g)$. Let z_0 be a zero of $f(z) - a$ of multiplicity s (respectively, a zero of $g(z) - a$ of multiplicity q). We denote by $\overline{N}_{f \geq p+1}(r, a; f | g \neq a)$ ($\overline{N}_{g \geq p+1}(r, a; g | f \neq a)$) the reduced counting functions of those a -points of f and g for which $s \geq p + 1$ and $q = 0$ ($q \geq p + 1$ and $s = 0$).

Definition 1.6 (see [5]). For $E_p(1; f) = E_p(1; g)$, let z_0 be a zero of $f(z) - 1$ with multiplicity $s (\geq 0)$ and a zero of $g(z) - 1$ with multiplicity $q (\geq 0)$. We denote by $\overline{N}_{\otimes}(r, 1; f, g)$ the reduced counting function of 1 points of f and g with $s \neq q$.

Clearly, we have

$$\begin{aligned} \overline{N}_{\otimes}(r, a; f, g) &= \overline{N}_L^{(p)}(r, a; f) + \overline{N}_L^{(p)}(r, a; g) \\ &\quad + \overline{N}_{f \geq p+1}(r, a; f | g \neq a) + \overline{N}_{g \geq p+1}(r, a; g | f \neq a) \\ &\leq \overline{N}(r, a; f | \geq p + 1) + \overline{N}(r, a; g | \geq p + 1). \end{aligned}$$

Definition 1.7 (see [12]). Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f | g = b)$ the counting function of those a -points of f , counted according to multiplicity, which are b -points of g .

Definition 1.8 (see [12]). Let $a, b_1, b_2, \dots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f | g \neq b_1, b_2, \dots, b_q)$ the counting function of those a -points of f , counted according to multiplicity, which are not the b_i -points of g for $i = 1, 2, \dots, q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in \mathbb{C} by

$$F = \frac{f^{n-2m}(f^{2m} + 2n(n-2m)f^m + n(n-2m)(n-m)^2)}{-c}, \quad (2.1)$$

$$G = \frac{g^{n-2m}(g^{2m} + n(n-2m)g^m + n(n-2m)(n-m)^2)}{-c}, \quad (2.2)$$

and let H be the function

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1} \right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1} \right). \quad (2.3)$$

Lemma 2.1 (see [14]). *If $E_p(1; F) = E_p(1; G)$ and $H \neq 0$, then*

$$N(r, 1; F | = 1) = N(r, 1; G | = 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.2 (see [15]). *Let f be a non-constant meromorphic function and let*

$$R(f) = \sum_{k=0}^n a_k f^k \bigg/ \sum_{j=0}^m b_j f^j$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f)$$

with $d = \max\{n, m\}$.

Lemma 2.3. *Let f and g be two non-constant meromorphic functions such that $E_p(S, f) = E_p(S, g)$, and let F, G be given by (2.1) and (2.2) with $H \neq 0$. Then*

$$\begin{aligned} N(r, H) &\leq \bar{N}(r, 0; f) + \bar{N}(r, 0; g) + \bar{N}(r, f) + \bar{N}(r, g) \\ &\quad + \bar{N}(r, 0; f^m + (n-m)(n-2m)) + \bar{N}(r, 0; g^m + (n-m)(n-2m)) \\ &\quad + N_{\otimes}(r, 1; F, G) + \bar{N}_0(r, 0; f') + \bar{N}_0(r, 0; g'), \end{aligned}$$

where $\bar{N}_0(r, 0; f')$ is the reduced counting function of those zeros of f' which are not the zeros of $f(f^m + (n-m)(n-2m))(F-1)$, and $\bar{N}_0(r, 0; g')$ is similarly defined.

Proof. Since $E_p(S, f) = E_p(S, g)$, from (2.1) and (2.2) we have $E_p(1, F) = E_p(1, G)$. Also

$$\begin{aligned} F' &= \frac{nf^{n-2m-1}(f^m + (n-m)(n-2m))^2 f'}{-c}, \\ G' &= \frac{ng^{n-2m-1}(g^m + (n-m)(n-2m))^2 g'}{-c}. \end{aligned}$$

Hence the result is obvious from equation (2.3). \square

Lemma 2.4 (see [5]). *Let f and g be two non-constant meromorphic functions such that $E_p(1; f) = E_p(1; g)$, where $1 \leq p < \infty$. Then*

$$\begin{aligned} & \overline{N}(r, 1; f) + \overline{N}(r, 1; g) - N(r, 1; f \mid = 1) \\ & + \left(\frac{p}{2} - \frac{1}{2}\right) \{ \overline{N}_{f \geq p+1}(r, 1; f \mid g \neq 1) + \overline{N}_{g \geq p+1}(r, 1; g \mid f \neq 1) \} \\ & + \left(p - \frac{1}{2}\right) \{ \overline{N}_L^{(p)}(r, 1; f) + \overline{N}_L^{(p)}(r, 1; g) \} \\ & \leq \frac{1}{2} [N(r, 1; f) + N(r, 1; g)]. \end{aligned}$$

Lemma 2.5. *Let F, G be given by (2.1). If $E_p(1; F) = E_p(1; G)$ and $\omega_1, \omega_2, \dots, \omega_n$ are the distinct roots of the equation*

$$z^n + n(n - 2m)z^{n-1} + bn(n - 2m)(n - m)^2 + c = 0$$

for $n \geq 3$. Then

$$\begin{aligned} \overline{N}_\otimes(r, 1; F, G) & \leq \frac{1}{p} [\overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \\ & \quad - N_\oplus(r, 0; f') - N_\oplus(r, 0; g')] + S(r, f) + S(r, g), \end{aligned}$$

where

$$N_\oplus(r, 0; f') = N(r, 0; f' \mid f \neq 0, \omega_1, \omega_2 \dots \omega_n)$$

and $N_\oplus(r, 0; g')$ is similarly defined.

Proof. Since

$$\begin{aligned} \overline{N}_\otimes(r, 1; F, G) & \leq \overline{N}(r, 1; F \mid \geq p + 1) + \overline{N}(r, 1; G \mid \geq p + 1) \\ & \leq \frac{1}{p} [N(r, 1; F) - \overline{N}(r, 1; F) + N(r, 1; G) - \overline{N}(r, 1; G)], \end{aligned}$$

the proof of the lemma can be carried out along the lines of the proof of Lemma 2.14 in [2]. □

Lemma 2.6. *Let F, G be given by (2.1). If $E_p(1; F) = E_p(1; G)$, then*

$$\begin{aligned} & (n + m)T(r, f) + (n + m)T(r, g) \\ & \leq 2\overline{N}(r, 0; f) + 2\overline{N}(r, \infty; f) + 2\overline{N}(r, 0; f^m + (n - m)(n - 2m)) \\ & \quad + 2\overline{N}(r, 0; g) + 2\overline{N}(r, 0; g^m + (n - m)(n - 2m)) + 2\overline{N}(r, \infty; g) \\ & \quad - \left(\frac{p}{2} - \frac{1}{2}\right) \{ \overline{N}_{f \geq p+1}(r, 1; f \mid g \neq 1) + \overline{N}_{g \geq p+1}(r, 1; g \mid f \neq 1) \} \\ & \quad - \left(p - \frac{1}{2}\right) \{ \overline{N}_L^{(p)}(r, 1; f) + \overline{N}_L^{(p)}(r, 1; g) \} + \overline{N}_\otimes(r, 1; F, G) \\ & \quad + \frac{n}{2} [T(r, f) + T(r, g)] + S(r, f) + S(r, g). \end{aligned}$$

Proof. By the second fundamental theorem, we have

$$\begin{aligned} & (n+m)T(r, f) + (n+m)T(r, g) \\ & \leq \overline{N}(r, 0; F-1) + \overline{N}(r, 0; f) + \overline{N}(r, \infty; f) \\ & \quad + \overline{N}(r, 0; f^m + (n-m)(n-2m)) + \overline{N}(r, 0; G-1) + \overline{N}(r, 0; g) \\ & \quad + \overline{N}(r, 0; g^m + (n-m)(n-2m)) + \overline{N}(r, \infty; g) \\ & \quad - \overline{N}_0(r, 0; f') - \overline{N}_0(r, 0; g') + S(r, f) + S(r, g). \end{aligned}$$

Now the result immediately follows from Lemmas 2.1–2.4. \square

Lemma 2.7. *Let $P(z)$ be defined by (1.1). Then*

- (i) $\beta_i \neq 0$,
- (ii) $P(z)$ is a critically injective polynomial for $n > 2$.

Proof. Clearly we have

$$P'(z) = nz^{n-2m-1}(z^m + (n-m)(n-2m))^2.$$

Let c_i to be the roots of the equation $z^m + (n-m)(n-2m) = 0$ for $i = 1, 2, \dots, m$, i.e., $c_i^m = -(n-m)(n-2m)$. Therefore,

$$c_i^{2m} + 2n(n-2m)c_i^m + n(n-2m)(n-m)^2 = 2(n-m)(n-2m)m^2.$$

- (i) Since $\gcd(m, n) = 1$ and $n > 2$, we have

$$\begin{aligned} \beta_i &= -(c_i^n + 2n(n-2m)c_i^{n-m} + n(n-2m)(n-m)^2c_i^{n-2m}) \\ &= -2c_i^{(n-2m)}(n-m)(n-2m)m^2 \neq 0. \end{aligned}$$

- (ii) We have

$$\begin{aligned} P(c_i) &= c_i^{(n-2m)}(c_i^{2m} + 2n(n-2m)c_i^m + n(n-2m)(n-m)^2) + c \\ &= 2c_i^{(n-2m)}(n-m)(n-2m)m^2 + c. \end{aligned}$$

So it is obvious that when $c_i \neq c_j$, $P(c_i) = P(c_j)$ implies

$$c_i^{(n-2m)} = c_j^{(n-2m)}. \quad (2.4)$$

Since $c_i^m = -(n-m)(n-2m) = c_j^m$, the equality (2.4) implies $c_i^n = c_j^n$, i.e., $(c_i/c_j)^n = 1$. Also from $c_i^m = c_j^m$, we have $(c_i/c_j)^m = 1$. Since $\gcd(m, n) = 1$, we get $c_i/c_j = 1$. Thus $c_i = c_j$ which is a contradiction. Hence $P(c_i) \neq P(c_j)$ for $c_i \neq c_j$. Also,

$$P(0) = c \neq 2c_i^{(n-2m)}(n-m)(n-2m)m^2 + c = P(c_i).$$

Thus $P(z)$ is critically injective. \square

Lemma 2.8 (see [7]). *Suppose that $P(z)$ is a monic polynomial without multiple zeros, whose derivatives have mutually distinct k zeros given by d_1, d_2, \dots, d_k with multiplicities q_1, q_2, \dots, q_k , respectively. Also suppose that $P(z)$ is critically injective. Then $P(z)$ is a uniqueness polynomial if and only if*

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{l=1}^k q_l.$$

In particular, the above inequality is always satisfied whenever $k \geq 4$. When $k = 3$ and $\max\{q_1, q_2, q_3\} \geq 2$, or when $k = 2$, $\min\{q_1, q_2\} \geq 2$, and $q_1 + q_2 \geq 5$, the above inequality also holds.

Lemma 2.9. *Let F, G be defined by (2.1) and (2.2). Then $F \equiv G$ implies $f \equiv g$ for $n \geq 2m + 4$.*

Proof. Since $F \equiv G$, we have $P(f) \equiv P(g)$. By Lemma 2.7 we know that $P(z)$ is critically injective. Also, we have

$$P'(z) = nz^{n-2m-1}(z^m + (n-m)(n-2m))^2,$$

which implies $k = m + 1$. Since $n - 2m - 1 \geq 3$, by Lemma 2.8 we get that $P(z)$ is a UPM and hence $f \equiv g$. \square

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Let us consider the following cases.

Case 1. Let $H \neq 0$.

Let f and g be two non-constant meromorphic functions such that $E_p(S, f) = E_p(S, g)$. Then from (2.1) and (2.2) we get $E_p(1, F) = E_p(1, G)$. Now we consider the following subcases.

Subcase 1.1. Let $p \geq 3$. Then, using Lemma 2.6 for $p \geq 3$, we get

$$\begin{aligned} & (n+m)T(r, f) + (n+m)T(r, g) \\ & \leq 2\overline{N}(r, 0; f) + 2\overline{N}(r, 0; g) + 2\overline{N}(r, f) + 2\overline{N}(r, g) \\ & \quad + 2\overline{N}(r, 0; f^m + (n-m)(n-2m)) + 2\overline{N}(r, 0; g^m + (n-m)(n-2m)) \\ & \quad + \frac{n}{2}(T(r, f) + T(r, g) + S(r, f) + S(r, g)) \end{aligned}$$

which is a contradiction for $n \geq 2m + 9$.

If f and g are entire functions, then setting $\overline{N}(r, \infty; f) = \overline{N}(r, \infty; g) = 0$, we get a contradiction for $n \geq 2m + 5$.

Subcase 1.2. Let $p = 2$. Then, by Lemmas 2.6 and 2.5, we have

$$\begin{aligned} & (n+m)T(r, f) + (n+m)T(r, g) \\ & \leq 2\overline{N}(r, 0; f) + 2\overline{N}(r, f) + 2\overline{N}(r, 0; f^m + (n-m)(n-2m)) \\ & \quad + 2\overline{N}(r, 0; g) + 2\overline{N}(r, 0; g^m + (n-m)(n-2m)) + 2\overline{N}(r, g) \end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \{ \overline{N}_{f \geq 3}(r, 1; f \mid g \neq 1) + \overline{N}_{g \geq 3}(r, 1; g \mid f \neq 1) \} \\
& -\frac{3}{2} \{ \overline{N}_L^2(r, 1; f) + \overline{N}_L^2(r, 1; g) \} + \overline{N}_\oplus(r, 1; F, G) \\
& + \frac{n}{2} [T(r, f) + T(r, g)] + S(r, f) + S(r, g) \\
& \leq (2m + 4 + \frac{n}{2}) [T(r, f) + T(r, g)] + \frac{1}{2} \overline{N}_\oplus(r, 1; F, G) + S(r, f) + S(r, g) \\
& \leq (2m + 4 + \frac{n}{2} + \frac{1}{2}) [T(r, f) + T(r, g)] + S(r, f) + S(r, g),
\end{aligned}$$

which is a contradiction for $n \geq 2m + 10$.

If f and g are entire functions, then setting $\overline{N}(r, \infty; f) = \overline{N}(r, \infty; g) = 0$ we get a contradiction for $n \geq 2m + 5$.

Subcase 1.3. Let $p = 1$. Then, again from Lemmas 2.6 and 2.5, we have

$$\begin{aligned}
& (n + m)T(r, f) + (n + m)T(r, g) \\
& \leq \overline{N}(r, 0; f) + 2\overline{N}(r, f) + 2\overline{N}(r, 0; f^m + (n - m)(n - 2m)) \\
& \quad + 2\overline{N}(r, 0; g) + 2\overline{N}(r, 0; g^m + (n - m)(n - 2m)) + 2\overline{N}(r, g) \\
& \quad - \frac{1}{2} \{ \overline{N}_L^1(r, 1; f) + \overline{N}_L^1(r, 1; g) \} + \overline{N}_\oplus(r, 1; F, G) \\
& \quad + \frac{n}{2} [T(r, f) + T(r, g)] + S(r, f) + S(r, g) \\
& \leq (2m + 4 + \frac{n}{2}) [T(r, f) + T(r, g)] + \overline{N}_\oplus(r, 1; F, G) + S(r, f) + S(r, g) \\
& \leq (2m + 4 + \frac{n}{2} + 2) [T(r, f) + T(r, g)] + S(r, f) + S(r, g),
\end{aligned}$$

which is a contradiction for $n \geq 2m + 13$.

If f and g are entire functions, then setting $\overline{N}(r, \infty; f) = \overline{N}(r, \infty; g) = 0$, we get a contradiction for $n \geq 2m + 7$.

Case 2. Let $H \equiv 0$. Then from (2.3) we get

$$\frac{1}{F - 1} \equiv \frac{A}{G - 1} + B, \quad (3.1)$$

where $A (\neq 0)$ and B are two constants. So in view of Lemma 2.2, from (3.1) we get

$$T(r, f) = T(r, g) + O(1). \quad (3.2)$$

Subcase 2.1. Suppose that $B \neq 0$. Then from (3.1) we get

$$F - 1 \equiv \frac{G - 1}{BG + A - B}. \quad (3.3)$$

Subcase 2.1.1. If $A - B \neq 0$, then noting that $\frac{B-A}{B} \neq 1$, from (3.3) we get

$$\overline{N}(r, \frac{B - A}{B}; G) = \overline{N}(r, \infty; F).$$

Therefore, in view of Lemma 2.2 and equation (3.2), using the second fundamental theorem, we have

$$\begin{aligned} nT(r, g) &\leq \bar{N}(r, 0; G) + \bar{N}(r, \infty; G) + \bar{N}\left(r, \frac{B-A}{B}; G\right) + S(r, G) \\ &\leq (2m+1)T(r, g) + \bar{N}(r, \infty; g) + \bar{N}(r, \infty; f) + S(r, g), \end{aligned}$$

which is a contradiction for $n \geq 2m + 4$.

Subcase 2.1.2. If $A - B = 0$, then from (3.3) we have

$$\begin{aligned} \frac{G-1}{F-1} &\equiv BG \\ &= B \frac{g^{n-2m}(g^{2m} + 2n(n-2m)g^m + n(n-2m)(n-m)^2)}{-c}, \end{aligned} \tag{3.4}$$

i.e., 0's of g and $(g^{2m} + 2n(n-2m)g^m + n(n-2m)(n-m)^2)$ are poles of F . It can be easily proved that all the zeros $\xi_i, i \in \{1, 2, \dots, 2m\}$, of $w^{2m} + 2n(n-2m)w^m + n(n-2m)(n-m)^2$ are simple. If each ξ_i -point of g is of multiplicity p , then it is a pole of F of multiplicity q for some $q \geq 1$. Thus from (3.4) we get $p = nq$, i.e., $p \geq n$. Similarly any zero of g of multiplicity r it is a pole of F of multiplicity s for some $s \geq 1$, i.e., $r(n-2m) = sn$. Hence $r = \frac{sn}{n-2m} \geq 2$ as $n > n-2m$. Now, using the second fundamental theorem, we get

$$\begin{aligned} (2m-1)T(r, g) &\leq \sum_{i=1}^{2m} \bar{N}(r, \xi_i; g) + \bar{N}(r, 0; g) + S(r, g) \\ &\leq \left(\frac{2m}{n} + \frac{1}{2}\right)T(r, g) + S(r, g), \end{aligned}$$

which is a contradiction for $n \geq 2m + 3$.

Subcase 2.2. Suppose that $B = 0$. Then from (3.1) we get

$$G - 1 = A(F - 1),$$

i.e.,

$$\begin{aligned} &g^n + 2n(n-2m)g^{n-m} + n(n-2m)(n-m)^2g^{n-2m} \\ &\equiv A \left(f^n + 2n(n-2m)f^{n-m} + n(n-2m)(n-m)^2f^{n-2m} + c \frac{A-1}{A} \right). \end{aligned} \tag{3.5}$$

Now we consider the following subcases.

Subcase 2.2.1. Let $A \neq 1$. Since $P(z)$ has no multiple zeros, $c \neq 0$. Hence $c \frac{A-1}{A} \neq 0$ and, as $\beta_i \neq 0$, we consider the following subcases.

Subcase 2.2.1.1. Suppose that

$$c \frac{A-1}{A} = \beta_i.$$

From Lemma 2.3, we know that

$$F' = n \frac{f^{n-2m-1}(f^m + (n-m)(n-2m))^2}{-c} f'.$$

Also by Lemma 2.7, we get $\beta_i \neq 0$ and that $P(z)$ is critically injective. Since any critically injective polynomial can have at most one multiple zero, we have

$$f^n + 2n(n-2m)f^{n-m} + n(n-2m)(n-m)^2 f^{n-2m} + \beta_i = (f - c_i)^3 \prod_{j=1}^{n-3} (f - \eta_j),$$

where η_j 's are $(n-3)$ distinct zeros of $z^n + 2n(n-2m)z^{n-m} + n(n-2m)(n-m)^2 z^{n-2m} + \beta_i$ such that $\eta_j \neq c_i, 0$. Then from (3.5), we get

$$\begin{aligned} g^{n-2m}(g^{2m} + 2n(n-2m)g^m + n(n-2m)(n-m)^2) \\ = A(f - c_i)^3 \prod_{j=1}^{n-3} (f - \eta_j). \end{aligned}$$

Therefore, using the second fundamental theorem and (3.2), we get

$$\begin{aligned} (n-3)T(r, f) &\leq \sum_{j=1}^{n-3} \bar{N}(r, \eta_j; f) + \bar{N}(r, c_i; f) + \bar{N}(r, \infty; f) + S(r, f) \\ &\leq (2m+1)T(r, g) + T(r, f) + S(r, f), \end{aligned}$$

which is a contradiction for $n \geq 2m + 6$.

Subcase 2.2.1.2. Suppose that

$$c \frac{(A-1)}{A} \neq \beta_i$$

for all $i \in \{1, 2, \dots, m\}$. So,

$$w^n + 2n(n-2m)w^{n-m} + n(n-2m)(n-m)^2 w^{n-2m} + c \frac{(A-1)}{A} = 0$$

has only simple roots, say α_i for $i = 1, 2, \dots, n$. Therefore, from (3.5) we have

$$g^{n-2m}(g^{2m} + ag^m + b) \equiv A \prod_{i=1}^n (f - \alpha_i).$$

Again, applying (3.2) and the second fundamental theorem, we get

$$\begin{aligned} (n-2)T(r, f) &\leq \sum_{i=1}^n \bar{N}(r, \alpha_i; f) + S(r, f) \\ &\leq (2m+1)T(r, g) + S(r, f), \end{aligned}$$

which is a contradiction for $n \geq 2m + 4$.

Subcase 2.2.2. If $A = 1$, then we have $F \equiv G$. Therefore, by Lemma 2.9, we have $f \equiv g$ for $n \geq 2m + 4$.

□

Acknowledgement

This research work is supported by the Council of Scientific and Industrial Research, Extramural Research Division, CSIR Complex, Pusa, New Delhi-110012, India, under the sanction project no. 25(0229)/14/EMR-II.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KALYANI, WEST BENGAL 741235, INDIA

E-mail address: abanerjee_kal@yahoo.co.in, abanerjeekal@gmail.com