Coefficient inequality for transforms of certain subclass of analytic functions

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Abstract. The objective of this paper is to obtain the best possible sharp upper bound for the second Hankel functional associated with the $k^{th}$ root transform $\left[ f(z^k) \right]^{1/k}$ of normalized analytic function $f(z)$ when it belongs to certain subclass of analytic functions, defined on the open unit disc in the complex plane using Toeplitz determinants.

1. Introduction

Let $A$ denote the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1.1)

defined in the open unit disc $E = \{ z : |z| < 1 \}$. Let $S$ be the subclass of $A$ consisting of univalent functions. In 1985, Louis de Branges de Bourcia [2] proved the Bieberbach conjecture, i.e., for an univalent function its $n^{th}$ coefficient is bounded by $n$. The bounds for the coefficients of these functions give the information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The $k^{th}$ root transform for the function $f$ given in (1.1) is defined as

$$F(z) := \left[ f(z^k) \right]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \hspace{1cm} (1.2)$$
Now, we introduce the Hankel determinant for the \( k \)th root transform for the function \( f \) given in (1.1), for \( q, n, k \in \mathbb{N} = \{1, 2, \ldots\} \), defined as

\[
\begin{vmatrix}
  b_{kn} & b_{kn+1} & \cdots & b_{k(n+q-2)+1} \\
  b_{kn+1} & b_{k(n+1)+1} & \cdots & b_{k(n+q-1)+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  b_{k(n+q-2)+1} & b_{k(n+q-1)+1} & \cdots & b_{k[n+2(q-1)-1]+1}
\end{vmatrix}^{1/k}.
\]

In particular, for \( k = 1 \) the above determinant reduces to the Hankel determinant defined by Pommerenke [9] for the function \( f \) given in (1.1), and this determinant has been investigated by several authors in the literature. In particular, for \( q = 2, n = 1, b_k = 1 \) and \( q = 2, n = 2, b_k = 1 \), the Hankel determinant simplifies, respectively, to

\[
\begin{vmatrix}
  b_{k+1} & b_{k+2} \\
  b_{k+2} & b_{k+3}
\end{vmatrix}^{1/k} = b_{k+1} - b_k^2
\]

and

\[
\begin{vmatrix}
  b_{2k+1} & b_{2k+2} \\
  b_{2k+2} & b_{2k+3}
\end{vmatrix}^{1/k} = b_{2k}b_{3k+1} - b_{2k+1}^2.
\]

For a family \( T \) of functions in \( S \), the more general problem of finding sharp estimates for the functional \( |a_3 - \mu a_2^2| \) (\( \mu \in \mathbb{R} \) or \( \mu \in \mathbb{C} \)) is popularly known as the Fekete–Szegö problem for \( T \). Ali et al. [1] obtained sharp bounds for the Fekete–Szegö functional denoted by \( |b_{2k+1} - \mu b_{k+1}^2| \) associated with the \( k \)th root transform \( [f(z^k)]^{1/k} \) of the function given in (1.1), belonging to certain subclasses of \( S \). We refer to \( |H_2(2)|^{1/k} \) as the second Hankel determinant for the \( k \)th root transform associated with the function \( f \). For our discussion in this paper, we consider the Hankel determinant given by \( |H_2(2)|^{1/k} \).

Motivated by the results obtained by Ali et al. [1], we obtain sharp upper bound to the functional \( |b_{k+1}b_{3k+1} - b_{2k+1}^2| \) for the \( k \)th root transform of the function \( f \) when it belongs to certain subclass denoted by \( Q(\alpha, \beta, \gamma) \) of \( S \), defined as follows.

**Definition 1.1.** A function \( f \in A \) is said to be in the class \( Q(\alpha, \beta, \gamma) \) with \( \alpha, \beta > 0 \) and \( 0 \leq \gamma < \alpha + \beta \leq 1 \), if it satisfies the condition

\[
\Re \left\{ \frac{\alpha f(z)}{z} + \beta f'(z) \right\} \geq \gamma, \quad z \in \mathbb{E}.
\]

This class was considered and studied by Wang et al. [12].

### 2. Preliminary results

Let \( \mathcal{P} \) denote the class of functions

\[
p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots = 1 + \sum_{n=1}^{\infty} c_n z^n
\]  

(2.1)
which are regular in the open unit disc $E$ and satisfy $\text{Re} \, p(z) > 0$ for any $z \in E$. Here $p(z)$ is called the Carathéodory function [3].

**Lemma 2.1** (see [9, 10]). If $p \in P$, then $|c_k| \leq 2$ for each $k \geq 1$, the inequality is sharp for the function $p_0(z) = (1 + z)/(1 - z)$.

**Lemma 2.2** (see [4]). The power series for $p(z)$ given in (2.1) converges in the open unit disc $E$ to a function in $P$ if and only if the Toeplitz determinants

$$D_n = \begin{vmatrix}
2 & c_1 & c_2 & \cdots & c_n \\
c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\
c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{-n} & c_{-n+1} & \cdots & 2
\end{vmatrix}, \quad n \in \mathbb{N}, \ c_{-k} = \bar{c}_k,$$

are all non-negative. They are strictly positive except $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k}z)$ with $\sum_{k=1}^{m} \rho_k = 1$, $t_k$ real, and $t_k \neq t_j$ for $k \neq j$. In this case, $D_n > 0$ for $n < (m - 1)$ and $D_n = 0$ for $n \geq m$.

This necessary and sufficient condition found in [4] is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2 for $n = 2$ and $n = 3$, we have, respectively,

$$2c_2 = c_1^2 + y(4 - c_1^2) \quad (2.2)$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta \quad (2.3)$$

for some complex valued $y$ with $|y| \leq 1$ and for some complex valued $\zeta$ with $|\zeta| \leq 1$. To obtain our result, we refer to the classical method initiated by Libera and Złotkiewicz [6], which has been used widely.

### 3. Main result

**Theorem 3.1.** If $f$ given by (1.1) belongs to $Q(\alpha, \beta, \gamma)$ with $\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$, then

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \left[ \frac{2(\alpha + \beta - \gamma)}{k(\alpha + 3\beta)} \right]^2$$

and the inequality is sharp.

**Proof.** Let $f \in Q(\alpha, \beta, \gamma)$. By virtue of Definition 1.1, there exists an analytic function $p \in P$ in the open unit disc $E$ with $p(0) = 1$ and $\text{Re} \, p(z) > 0$ such that

$$\frac{\alpha f(z) + \beta zf'(z) - \gamma z}{(\alpha + \beta - \gamma)z} = p(z). \quad (3.1)$$
Replacing \( f(z) \), \( f'(z) \) and \( p(z) \) with their equivalent series expressions in the relation (3.1), we have

\[
\alpha \left\{ z + \sum_{n=2}^{\infty} a_n z^n \right\} + \beta z \left\{ 1 + \sum_{n=2}^{\infty} n a_n z^{n-1} \right\} - \gamma z
= (\alpha + \beta - \gamma) z \left\{ 1 + \sum_{n=1}^{\infty} c_n z^n \right\}.
\]

Upon simplification, we obtain

\[
(\alpha + 2\beta)a_2 + (\alpha + 3\beta)a_3 + (\alpha + 4\beta)a_4 z^2 + \ldots
= (\alpha + \beta - \gamma) (c_1 + c_2 z + c_3 z^2 + \ldots).
\] (3.2)

Equating the coefficients of like powers of \( z^0 \), \( z^1 \) and \( z^2 \), respectively, on both sides of (3.2), we get

\[
a_2 = \frac{\alpha + \beta - \gamma}{\alpha + 2\beta} c_1, \quad a_3 = \frac{\alpha + \beta - \gamma}{\alpha + 3\beta} c_2, \quad a_4 = \frac{\alpha + \beta - \gamma}{\alpha + 4\beta} c_3.
\] (3.3)

For a function \( f \) given by (1.1), a computation shows that

\[
\left[ f(z^k) \right]^{\frac{1}{k}} = \left[ z^k + \sum_{n=2}^{\infty} a_n z^{nk} \right]^{\frac{1}{k}}
= z + \frac{1}{k} a_2 z^{k+1} + \left\{ \frac{1}{k} a_3 + \frac{1 - k}{2k^2} a_2^2 \right\} z^{2k+1}
+ \left\{ \frac{1}{k} a_4 + \frac{1 - k}{k^2} a_2 a_3 + \frac{(1 - k)(1 - 2k)}{6k^3} a_2^3 \right\} z^{3k+1} + \ldots
\] (3.4)

The expressions (1.2) and (3.4) yield

\[
b_{k+1} = \frac{1}{k} a_2, \quad b_{2k+1} = \frac{1}{k} a_3 + \frac{1 - k}{2k^2} a_2^2,
\]

\[
b_{3k+1} = \frac{1}{k} a_4 + \frac{1 - k}{k^2} a_2 a_3 + \frac{(1 - k)(1 - 2k)}{6k^3} a_2^3.
\] (3.5)

Simplifying the relations (3.3) and (3.5), we get

\[
b_{k+1} = \frac{\alpha + \beta - \gamma}{k(\alpha + 2\beta)} c_1,
\]

\[
b_{2k+1} = \frac{\alpha + \beta - \gamma}{k} \left[ \frac{1}{(\alpha + 3\beta)} c_2 + \frac{(1 - k)(\alpha + \beta - \gamma)}{2k(\alpha + 2\beta)^2} c_1^2 \right],
\]

\[
b_{3k+1} = \frac{\alpha + \beta - \gamma}{k} \left[ \frac{1}{(\alpha + 4\beta)} c_3 + \frac{(1 - k)(\alpha + \beta - \gamma)}{k(\alpha + 2\beta)(\alpha + 3\beta)} c_1 c_2
+ \frac{(1 - k)(1 - 2k)(\alpha + \beta - \gamma)^2}{6k^2(\alpha + 2\beta)^3} c_1^3 \right].
\] (3.6)
Substituting the values of $b_{k+1}, b_{2k+1}$ and $b_{3k+1}$ from (3.6) in the second Hankel determinant to the $k^{th}$ transform for the function $f \in Q(\alpha, \beta, \gamma)$, which simplifies to give

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| = \frac{(\alpha + \beta - \gamma)^2}{12k^4(\alpha + 2\beta)^4(\alpha + 3\beta)^2(\alpha + 4\beta)} \times 12k^2(\alpha + \beta)^3(\alpha + 3\beta)^2c_1c_3 - 12k^2(\alpha + 2\beta)^4(\alpha + 4\beta)c_2^2 + (k^2 - 1)(\alpha + \beta - \gamma)^2(\alpha + 3\beta)^2(\alpha + 4\beta)c_1^2. \quad (3.7)$$

The above expression is equivalent to

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| = t |d_1c_1c_3 + d_2c_2^2 + d_3c_3^4|, \quad (3.8)$$

where

$$t = \frac{(\alpha + \beta - \gamma)^2}{12k^4(\alpha + 2\beta)^4(\alpha + 3\beta)^2(\alpha + 4\beta)} \quad (3.9)$$

and

$$d_1 = 12k^2(\alpha + 2\beta)^3(\alpha + 3\beta)^2, \quad d_2 = 12k^2(\alpha + 2\beta)^4(\alpha + 4\beta), \quad d_3 = (k^2 - 1)(\alpha + \beta - \gamma)^2(\alpha + 3\beta)^2(\alpha + 4\beta). \quad (3.10)$$

Substituting the values of $c_2$ and $c_3$ from (2.2) and (2.3), respectively, from Lemma 2.2 on the right-hand side of (3.8), we have

$$|d_1c_1c_3 + d_2c_2^2 + d_3c_3^4| = \frac{1}{4}d_1c_1\{c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta\} + \frac{1}{4}d_2\{c_2^2 + y(4 - c_1^2)\}^2 + d_3c_3^4. \quad (3.11)$$

Using the triangle inequality and the fact that $|\zeta| < 1$, after simplifying we get

$$4 |d_1c_1c_3 + d_2c_2^2 + d_3c_3^4| \leq |(d_1 + d_2 + 4d_3)c_1^4 + 2d_1c_1(4 - c_1^2) + 2(d_1 + d_2)c_1^2(4 - c_1^2)|y| - \{(d_1 + d_2)c_1^2 + 2d_1c_1 - 4d_2\} (4 - c_1^2)|y|^2\}. \quad (3.12)$$

Using the values of $d_1, d_2$ and $d_3$ from (3.10), we can write

$$(d_1 + d_2)c_1^2 + 2d_1c_1 - 4d_2 = 12k^2(\alpha + 2\beta)^3 \times \{\beta^2c_1^2 + 2(\alpha + 3\beta)c_1 + 4(\alpha + 2\beta)(\alpha + 4\beta)\}. \quad (3.12)$$

Consider

$$\beta^2c_1^2 + 2(\alpha + 3\beta)c_1 + 4(\alpha + 2\beta)(\alpha + 4\beta)$$

$$= \beta^2 \left\{ c_1 + \frac{1}{\beta^2}(\alpha + 3\beta)^2 \right\} - \left\{ \frac{\sqrt{\alpha^4 + 49\beta^4 + 50\alpha^2\beta^2 + 84\alpha\beta^3 + 12\alpha^3\beta}}{\beta^4} \right\}^2.$$


where

\[ F(c, \mu) = \left\{ 12k^2(\alpha + 2\beta)^3 \beta^2 - 4(k^2 - 1)(\alpha + \beta - \gamma)^2(\alpha + 3\beta)^2(\alpha + 4\beta) \right\} c^4 + 24k^2(\alpha + 2\beta)^3 \left\{ (\alpha + 3\beta)^2 c + \beta^2 c^2 \mu \right\} (4 - c^2) + 12k^2(\alpha + 2\beta)^3 \left\{ \beta^2 c^2 - 2(\alpha + 3\beta)^2 c \right\} + 4(\alpha + 2\beta)(\alpha + 4\beta) \right\} (4 - c^2) \mu^2. \]
Next, we maximize the function $F(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ in (3.16) partially with respect to $\mu$, we get

$$\frac{\partial F}{\partial \mu} = 24k^2(\alpha + 2\beta)^3 [\beta^2 c^2 + \left\{\beta^2 c^2 - 2(\alpha + 3\beta)^2 c + 4(\alpha + 2\beta)(\alpha + 4\beta)\right\} \mu] (4 - c^2). \quad (3.17)$$

For $0 < \mu < 1$, for fixed $c$ with $0 < c < 2$ and $\alpha, \beta > 0$, from (3.17) we observe that $\frac{\partial F}{\partial \mu} > 0$. Consequently, $F(c, \mu)$ becomes an increasing function of $\mu$ and, hence, $F(c, \mu)$ cannot have a maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. Further, for fixed $c \in [0, 2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c). \quad (3.18)$$

Simplifying the relations (3.16) and (3.18), we obtain

$$G(c) = -4\left\{(k^2 - 1)(\alpha + \beta - \gamma)^2(\alpha + 3\beta)^2(\alpha + 4\beta) + 6k^2\beta^2(\alpha + 2\beta)^3\right\} c^4 - 48k^2(\alpha + 2\beta)^3(\alpha^2 + 6\alpha\beta + 6\beta^2)c^2 + 192k^2(\alpha + 2\beta)^4(\alpha + 4\beta), \quad (3.19)$$

and, consequently,

$$G'(c) = -16\left\{(k^2 - 1)(\alpha + \beta - \gamma)^2(\alpha + 3\beta)^2(\alpha + 4\beta) + 6k^2\beta^2(\alpha + 2\beta)^3\right\} c^3 - 96k^2(\alpha + 2\beta)^3(\alpha^2 + 6\alpha\beta + 6\beta^2)c. \quad (3.20)$$

From the expression (3.20), we observe that $G'(c) \leq 0$ for all values of $c \in [0, 2]$ and for fixed values of $\alpha, \beta > 0$, where $0 \leq \gamma < \alpha + \beta \leq 1$. Therefore, $G(c)$ becomes a monotonically decreasing function of $c$ in the interval $[0, 2]$ and hence it attains the maximum value at $c = 0$ only. From (3.19), the maximum value of $G(c)$ is given by

$$\max_{0 \leq c \leq 2} G(c) = G(0) = 192k^2(\alpha + 2\beta)^4(\alpha + 4\beta). \quad (3.21)$$

Considering, only the maximum value of $G(c)$ at $c = 0$, from the relations (3.15) and (3.21), after simplifying, we get

$$|d_1c_1c_3 + d_2c_2^2 + d_3c_1^3| \leq 48k^2(\alpha + 2\beta)^4(\alpha + 4\beta). \quad (3.22)$$

Simplifying the expressions (3.8) and (3.22) together with (3.9), we obtain

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \left[\frac{2(\alpha + \beta - \gamma)}{k(\alpha + 3\beta)}\right]^2. \quad (3.23)$$

If we set $c_1 = c = 0$ and select $y = 1$ in (2.2) and (2.3), we find that $c_2 = 2$ and $c_3 = 0$. Using these values in (3.22), we observe that equality is
attained, which shows that our result is sharp. For these values, we derive the extremal function from (2.1), given by
\[
\frac{\alpha f(z)}{z} + \beta f'(z) - \gamma = \frac{\alpha f(z) + \beta z f'(z) - \gamma z}{(\alpha + \beta - \gamma)z} = 1 + 2z^2 + 2z^4 - \cdots = \frac{1 - z^2}{1 + z^2}.
\]
This completes the proof of our theorem. □

**Remark 3.2.** For the choice of \(\alpha = (1 - \sigma), \beta = \sigma\) and \(\gamma = 0\), we get
\[(\alpha, \beta, \gamma) = ((1 - \sigma), \sigma, 0),\]
for which, from (3.23), upon simplification, we obtain
\[|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \frac{4}{(1 + 2\sigma)^2}, \quad 0 \leq \sigma \leq 1.\]
This result is a special case of that of Murugusundaramoorthy and Magesh [7].

**Remark 3.3.** Selecting \(k = 1, \alpha = 0, \beta = 1\) and \(\gamma = 0\) in (3.23), we obtain
\[|b_2b_4 - b_3^2| \leq \frac{4}{9}.\]
This result coincides with that of Janteng et al. [5].

**Remark 3.4.** Choosing \(k = 1\) in (3.23), we obtain
\[|b_2b_4 - b_3^2| \leq \frac{4(\alpha + \beta - \gamma)^2}{(\alpha + 3\beta)^2}.\]
This result coincides with that of Vamshee Krishna and RamReddy [11].

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