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Coefficient inequality for transforms of certain subclass of analytic functions

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Abstract. The objective of this paper is to obtain the best possible sharp upper bound for the second Hankel functional associated with the k^{th} root transform $\left[f(z^k)\right]^{1/k}$ of normalized analytic function $f(z)$ when it belongs to certain subclass of analytic functions, defined on the open unit disc in the complex plane using Toeplitz determinants.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

defined in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia [2] proved the Bieberbach conjecture, i.e., for an univalent function its n^{th} coefficient is bounded by n. The bounds for the coefficients of these functions give the information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The k^{th} root transform for the function f given in (1.1) is defined as

$$F(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.2}$$

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Now, we introduce the Hankel determinant for the k^{th} root transform for the function f given in (1.1), for $q, n, k \in \mathbb{N} = \{1, 2, \dots\}$, defined as

$$|H_q(n)|^{\frac{1}{k}} = \begin{vmatrix} b_{kn} & b_{kn+1} & \cdots & b_{k(n+q-2)+1} \\ b_{kn+1} & b_{k(n+1)+1} & \cdots & b_{k(n+q-1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k(n+q-2)+1} & b_{k(n+q-1)+1} & \cdots & b_{k[n+2(q-1)-1]+1} \end{vmatrix}.$$

In particular, for $k = 1$ the above determinant reduces to the Hankel determinant defined by Pommerenke [9] for the function f given in (1.1), and this determinant has been investigated by several authors in the literature. In particular, for $q = 2, n = 1, b_k = 1$ and $q = 2, n = 2, b_k = 1$, the Hankel determinant simplifies, respectively, to

$$|H_2(1)|^{\frac{1}{k}} = \begin{vmatrix} b_k & b_{k+1} \\ b_{k+1} & b_{2k+1} \end{vmatrix} = b_{2k+1} - b_{k+1}^2$$

and

$$|H_2(2)|^{\frac{1}{k}} = \begin{vmatrix} b_{2k} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix} = b_{2k}b_{3k+1} - b_{2k+1}^2.$$

For a family $\mathcal{T}$ of functions in S, the more general problem of finding sharp estimates for the functional $|a_3 - \mu a_2^2|$ ($\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$) is popularly known as the Fekete–Szegö problem for $\mathcal{T}$. Ali et al. [1] obtained sharp bounds for the Fekete–Szegö functional denoted by $|b_{2k+1} - \mu b_{k+1}^2|$ associated with the k^{th} root transform $\left[f(z^k)\right]^{1/k}$ of the function given in (1.1), belonging to certain subclasses of S. We refer to $|H_2(2)|^{1/k}$ as the second Hankel determinant for the k^{th} root transform associated with the function f. For our discussion in this paper, we consider the Hankel determinant given by $|H_2(2)|^{1/k}$. Motivated by the results obtained by Ali et al. [1], we obtain sharp upper bound to the functional $|b_{k+1}b_{3k+1} - b_{2k+1}^2|$ for the k^{th} root transform of the function f when it belongs to certain subclass denoted by $Q(\alpha, \beta, \gamma)$ of S, defined as follows.

Definition 1.1. A function $f \in A$ is said to be in the class $Q(\alpha, \beta, \gamma)$ with α, $\beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$, if it satisfies the condition

$$\operatorname{Re}\left\{\alpha \frac{f(z)}{z} + \beta f'(z)\right\} \geq \gamma, \quad z \in E.$$

This class was considered and studied by Wang et al. [12].

2. Preliminary results

Let $\mathscr{P}$ denote the class of functions

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\operatorname{Re} p(z) > 0$ for any $z \in E$. Here $p(z)$ is called the Carathéodory function [3].

Lemma 2.1 (see [9, 10])**.** *If $p \in \mathscr{P}$, then $|c_k| \leq 2$ for each $k \geq 1$, the inequality is sharp for the function $p_0(z) = (1+z)/(1-z)$.*

Lemma 2.2 (see [4])**.** *The power series for $p(z)$ given in* (2.1) *converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n \in \mathbb{N}, \; c_{-k} = \bar{c}_k,$$

are all non-negative. They are strictly positive except $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$ with $\sum_{k=1}^{m} \rho_k = 1$, t_k real, and $t_k \neq t_j$ for $k \neq j$. In this case, $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [4] is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2 for $n = 2$ and $n = 3$, we have, respectively,

$$2c_2 = c_1^2 + y(4 - c_1^2) \tag{2.2}$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta \tag{2.3}$$

for some complex valued y with $|y| \leq 1$ and for some complex valued ζ with $|\zeta| \leq 1$. To obtain our result, we refer to the classical method initiated by Libera and Złotkiewicz [6], which has been used widely.

3. Main result

Theorem 3.1. *If f given by* (1.1) *belongs to $Q(\alpha, \beta, \gamma)$ with $\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$), then*

$$\left| b_{k+1} b_{3k+1} - b_{2k+1}^2 \right| \leq \left[\frac{2(\alpha + \beta - \gamma)}{k(\alpha + 3\beta)} \right]^2$$

and the inequality is sharp.

Proof. Let $f \in Q(\alpha, \beta, \gamma)$. By virtue of Definition 1.1, there exists an analytic function $p \in \mathscr{P}$ in the open unit disc E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ such that

$$\frac{\alpha f(z) + \beta z f'(z) - \gamma z}{(\alpha + \beta - \gamma) z} = p(z). \tag{3.1}$$

Replacing $f(z)$, $f'(z)$ and $p(z)$ with their equivalent series expressions in the relation (3.1), we have

$$\alpha\left\{z+\sum_{n=2}^{\infty}a_nz^n\right\}+\beta z\left\{1+\sum_{n=2}^{\infty}na_nz^{n-1}\right\}-\gamma z$$
$$=(\alpha+\beta-\gamma)z\left\{1+\sum_{n=1}^{\infty}c_nz^n\right\}.$$

Upon simplification, we obtain

$$\begin{aligned}&(\alpha+2\beta)a_2+(\alpha+3\beta)a_3z+(\alpha+4\beta)a_4z^2+\dots\\&\qquad=(\alpha+\beta-\gamma)\left(c_1+c_2z+c_3z^2+\dots\right).\end{aligned}\tag{3.2}$$

Equating the coefficients of like powers of z^0, z^1 and z^2, respectively, on both sides of (3.2), we get

$$a_2=\frac{\alpha+\beta-\gamma}{\alpha+2\beta}c_1,\quad a_3=\frac{\alpha+\beta-\gamma}{\alpha+3\beta}c_2,\quad a_4=\frac{\alpha+\beta-\gamma}{\alpha+4\beta}c_3.\tag{3.3}$$

For a function f given by (1.1), a computation shows that

$$\begin{aligned}\left[f(z^k)\right]^{\frac{1}{k}}&=\left[z^k+\sum_{n=2}^{\infty}a_nz^{nk}\right]^{\frac{1}{k}}\\&=z+\frac{1}{k}a_2z^{k+1}+\left\{\frac{1}{k}a_3+\frac{1-k}{2k^2}a_2^2\right\}z^{2k+1}\\&\quad+\left\{\frac{1}{k}a_4+\frac{1-k}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{6k^3}a_2^3\right\}z^{3k+1}+\dots.\end{aligned}\tag{3.4}$$

The expressions (1.2) and (3.4) yield

$$\begin{aligned}b_{k+1}&=\frac{1}{k}a_2,\quad b_{2k+1}=\frac{1}{k}a_3+\frac{1-k}{2k^2}a_2^2,\\b_{3k+1}&=\frac{1}{k}a_4+\frac{1-k}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{6k^3}a_2^3.\end{aligned}\tag{3.5}$$

Simplifying the relations (3.3) and (3.5), we get

$$\begin{aligned}b_{k+1}&=\frac{\alpha+\beta-\gamma}{k(\alpha+2\beta)}c_1,\\b_{2k+1}&=\frac{\alpha+\beta-\gamma}{k}\left[\frac{1}{(\alpha+3\beta)}c_2+\frac{(1-k)(\alpha+\beta-\gamma)}{2k(\alpha+2\beta)^2}c_1^2\right],\\b_{3k+1}&=\frac{\alpha+\beta-\gamma}{k}\left[\frac{1}{(\alpha+4\beta)}c_3+\frac{(1-k)(\alpha+\beta-\gamma)}{k(\alpha+2\beta)(\alpha+3\beta)}c_1c_2\right.\\&\qquad\left.+\frac{(1-k)(1-2k)(\alpha+\beta-\gamma)^2}{6k^2(\alpha+2\beta)^3}c_1^3\right].\end{aligned}\tag{3.6}$$

Substituting the values of b_{k+1}, b_{2k+1} and b_{3k+1} from (3.6) in the second Hankel determinant to the k^{th} transform for the function $f \in Q(\alpha, \beta, \gamma)$, which simplifies to give

$$\begin{aligned} |b_{k+1}b_{3k+1} - b_{2k+1}^2| &= \frac{(\alpha+\beta-\gamma)^2}{12k^4(\alpha+2\beta)^4(\alpha+3\beta)^2(\alpha+4\beta)} \\ &\times \Big|12k^2(\alpha+\beta)^3(\alpha+3\beta)^2c_1c_3 - 12k^2(\alpha+2\beta)^4(\alpha+4\beta)c_2^2 \\ &+ (k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)c_1^4\Big|. \end{aligned} \tag{3.7}$$

The above expression is equivalent to

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| = t\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right|, \tag{3.8}$$

where

$$t = \frac{(\alpha+\beta-\gamma)^2}{12k^4(\alpha+2\beta)^4(\alpha+3\beta)^2(\alpha+4\beta)} \tag{3.9}$$

and

$$\begin{aligned} d_1 &= 12k^2(\alpha+2\beta)^3(\alpha+3\beta)^2, \\ d_2 &= 12k^2(\alpha+2\beta)^4(\alpha+4\beta), \\ d_3 &= (k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta). \end{aligned} \tag{3.10}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.3), respectively, from Lemma 2.2 on the right-hand side of (3.8), we have

$$\begin{aligned} \left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| &= \Big|\frac{1}{4}d_1c_1\{c_1^3 + 2c_1(4-c_1^2)y - c_1(4-c_1^2)y^2 \\ &+ 2(4-c_1^2)(1-|y|^2)\zeta\} + \frac{1}{4}d_2\{c_1^2 + y(4-c_1^2)\}^2 + d_3c_1^4\Big|. \end{aligned}$$

Using the triangle inequality and the fact that $|\zeta| < 1$, after simplifying we get

$$\begin{aligned} 4\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| \leq &\Big|(d_1+d_2+4d_3)c_1^4 \\ &+2d_1c_1(4-c_1^2) + 2(d_1+d_2)c_1^2(4-c_1^2)|y| \\ &- \left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\}(4-c_1^2)|y|^2\Big|. \end{aligned} \tag{3.11}$$

Using the values of d_1, d_2 and d_3 from (3.10), we can write

$$\begin{aligned} (d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2 &= 12k^2(\alpha+2\beta)^3 \\ &\times \left\{\beta^2c_1^2 + 2(\alpha+3\beta)^2c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\}. \end{aligned} \tag{3.12}$$

Consider

$$\beta^2c_1^2 + 2(\alpha+3\beta)^2c_1 + 4(\alpha+2\beta)(\alpha+4\beta)$$

$$= \beta^2\left[\left\{c_1 + \frac{(\alpha+3\beta)^2}{\beta^2}\right\}^2 - \left\{\sqrt{\frac{\alpha^4+49\beta^4+50\alpha^2\beta^2+84\alpha\beta^3+12\alpha^3\beta}{\beta^4}}\right\}^2\right]$$

$$= \beta^2 \left[c_1 + \left\{ \frac{(\alpha+3\beta)^2}{\beta^2} + \sqrt{\frac{\alpha^4+49\beta^4+50\alpha^2\beta^2+84\alpha\beta^3+12\alpha^3\beta}{\beta^4}} \right\} \right]$$
$$\times \left[c_1 + \left\{ \frac{(\alpha+3\beta)^2}{\beta^2} - \sqrt{\frac{\alpha^4+49\beta^4+50\alpha^2\beta^2+84\alpha\beta^3+12\alpha^3\beta}{\beta^4}} \right\} \right].$$

Since $c_1 \in [0,2]$, noting that $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a,b \geq 0$ on the right-hand side of above expression, we have

$$\begin{aligned} &\beta^2 c_1^2 + 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta) \\ &\qquad \geq \beta^2 c_1^2 - 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta). \end{aligned} \tag{3.13}$$

From the relations (3.12) and (3.13), we get

$$\begin{aligned} &-\left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\} \leq -12k^2(\alpha+2\beta)^3 \\ &\qquad \times \left\{\beta^2 c_1^2 - 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\}. \end{aligned} \tag{3.14}$$

Substituting the calculated values from (3.10) and (3.14) on the right-hand side of (3.11), we have

$$\begin{aligned} 4|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4| \leq \Big| &\big[12k^2(\alpha+2\beta)^3\beta^2 \\ &- 4(k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)\big]c_1^4 \\ &+ 24k^2(\alpha+2\beta)^3\left\{(\alpha+3\beta)^2c_1 + \beta^2c_1^2|y|\right\}(4-c_1^2) \\ &- 12k^2(\alpha+2\beta)^3\left\{\beta^2c_1^2 - 2(\alpha+3\beta)^2c_1\right. \\ &\left.+4(\alpha+2\beta)(\alpha+4\beta)\right\}(4-c_1^2)|y|^2\Big|. \end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying the triangle inequality and replacing $|y|$ by μ on the right-hand side of the above inequality, we obtain

$$4|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4| \leq F(c,\mu), \tag{3.15}$$

where

$$\begin{aligned} F(c,\mu) = &\left\{12k^2(\alpha+2\beta)^3\beta^2\right. \\ &\left.- 4(k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)\right\}c^4 \\ &+ 24k^2(\alpha+2\beta)^3\left\{(\alpha+3\beta)^2c + \beta^2c^2\mu\right\}(4-c^2) \\ &+ 12k^2(\alpha+2\beta)^3\left\{\beta^2c^2 - 2(\alpha+3\beta)^2c\right. \\ &\left.+4(\alpha+2\beta)(\alpha+4\beta)\right\}(4-c^2)\mu^2. \end{aligned} \tag{3.16}$$

Next, we maximize the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (3.16) partially with respect to μ, we get

$$\begin{aligned}\frac{\partial F}{\partial \mu} &= 24k^2(\alpha+2\beta)^3\left[\beta^2c^2\right.\\ &\quad\left.+\left\{\beta^2c^2-2(\alpha+3\beta)^2c+4(\alpha+2\beta)(\alpha+4\beta)\right\}\mu\right](4-c^2).\end{aligned}\tag{3.17}$$

For $0<\mu<1$, for fixed c with $0<c<2$ and $\alpha,\beta>0$, from (3.17) we observe that $\frac{\partial F}{\partial \mu}>0$. Consequently, $F(c,\mu)$ becomes an increasing function of μ and, hence, $F(c,\mu)$ cannot have a maximum value at any point in the interior of the closed region $[0,2]\times[0,1]$. Further, for fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le1} F(c,\mu)=F(c,1)=G(c).\tag{3.18}$$

Simplifying the relations (3.16) and (3.18), we obtain

$$\begin{aligned}G(c) &= -4\big\{(k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)\\ &\quad+6k^2\beta^2(\alpha+2\beta)^3\big\}c^4-48k^2(\alpha+2\beta)^3(\alpha^2+6\alpha\beta+6\beta^2)c^2\\ &\quad+192k^2(\alpha+2\beta)^4(\alpha+4\beta),\end{aligned}\tag{3.19}$$

and, consequently,

$$\begin{aligned}G'(c) &= -16\big\{(k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)\\ &\quad+6k^2\beta^2(\alpha+2\beta)^3\big\}c^3-96k^2(\alpha+2\beta)^3(\alpha^2+6\alpha\beta+6\beta^2)c.\end{aligned}\tag{3.20}$$

From the expression (3.20), we observe that $G'(c)\le 0$ for all values of $c\in[0,2]$ and for fixed values of α, $\beta>0$, where $0\le\gamma<\alpha+\beta\le1$. Therefore, $G(c)$ becomes a monotonically decreasing function of c in the interval $[0,2]$ and hence it attains the maximum value at $c=0$ only. From (3.19), the maximum value of $G(c)$ is given by

$$\max_{0\le c\le 2} G(c)=G(0)=192k^2(\alpha+2\beta)^4(\alpha+4\beta).\tag{3.21}$$

Considering, only the maximum value of $G(c)$ at $c=0$, from the relations (3.15) and (3.21), after simplifying, we get

$$|d_1c_1c_3+d_2c_2^2+d_3c_1^4|\le 48k^2(\alpha+2\beta)^4(\alpha+4\beta).\tag{3.22}$$

Simplifying the expressions (3.8) and (3.22) together with (3.9), we obtain

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|\le\left[\frac{2(\alpha+\beta-\gamma)}{k(\alpha+3\beta)}\right]^2.\tag{3.23}$$

If we set $c_1=c=0$ and select $y=1$ in (2.2) and (2.3), we find that $c_2=2$ and $c_3=0$. Using these values in (3.22), we observe that equality is attained, which shows that our result is sharp. For these values, we derive the extremal function from (2.1), given by

$$\alpha\frac{f(z)}{z}+\beta f'(z)-\gamma=\frac{\alpha f(z)+\beta zf'(z)-\gamma z}{(\alpha+\beta-\gamma)z}=1+2z^2+2z^4-\cdots=\frac{1-z^2}{1+z^2}.$$

This completes the proof of our theorem. □

Remark 3.2. For the choice of $\alpha=(1-\sigma)$, $\beta=\sigma$ and $\gamma=0$, we get

$$(\alpha,\beta,\gamma)=((1-\sigma),\sigma,0),$$

for which, from (3.23), upon simplification, we obtain

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|\leq\frac{4}{(1+2\sigma)^2},\quad 0\leq\sigma\leq 1.$$

This result is a special case of that of Murugusundaramoorthy and Magesh [7].

Remark 3.3. Selecting $k=1$, $\alpha=0$, $\beta=1$ and $\gamma=0$ in (3.23), we obtain

$$|b_2b_4-b_3^2|\leq\frac{4}{9}.$$

This result coincides with that of Janteng et al. [5].

Remark 3.4. Choosing $k=1$ in (3.23), we obtain

$$|b_2b_4-b_3^2|\leq\frac{4(\alpha+\beta-\gamma)^2}{(\alpha+3\beta)^2}.$$

This result coincides with that of Vamshee Krishna and RamReddy [11].

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