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Coefficient inequality for transforms of certain subclass of analytic functions

T. RamReddy, D. Shalini, D. Vamshee Krishna, and B. Venkateswarlu

Abstract. The objective of this paper is to obtain the best possible sharp upper bound for the second Hankel functional associated with the k^{th} root transform $\left[f(z^k)\right]^{1/k}$ of normalized analytic function $f(z)$ when it belongs to certain subclass of analytic functions, defined on the open unit disc in the complex plane using Toeplitz determinants.

1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

defined in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia [2] proved the Bieberbach conjecture, i.e., for an univalent function its n^{th} coefficient is bounded by n. The bounds for the coefficients of these functions give the information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The k^{th} root transform for the function f given in (1.1) is defined as

$$F(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.2}$$

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Corresponding author: D. Vamshee Krishna

Now, we introduce the Hankel determinant for the k^{th} root transform for the function f given in (1.1), for $q, n, k \in \mathbb{N} = \{1, 2, \dots\}$, defined as

$$|H_q(n)|^{\frac{1}{k}} = \begin{vmatrix} b_{kn} & b_{kn+1} & \cdots & b_{k(n+q-2)+1} \\ b_{kn+1} & b_{k(n+1)+1} & \cdots & b_{k(n+q-1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k(n+q-2)+1} & b_{k(n+q-1)+1} & \cdots & b_{k[n+2(q-1)-1]+1} \end{vmatrix}.$$

In particular, for $k = 1$ the above determinant reduces to the Hankel determinant defined by Pommerenke [9] for the function f given in (1.1), and this determinant has been investigated by several authors in the literature. In particular, for $q = 2, n = 1, b_k = 1$ and $q = 2, n = 2, b_k = 1$, the Hankel determinant simplifies, respectively, to

$$|H_2(1)|^{\frac{1}{k}} = \begin{vmatrix} b_k & b_{k+1} \\ b_{k+1} & b_{2k+1} \end{vmatrix} = b_{2k+1} - b_{k+1}^2$$

and

$$|H_2(2)|^{\frac{1}{k}} = \begin{vmatrix} b_{2k} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix} = b_{2k}b_{3k+1} - b_{2k+1}^2.$$

For a family $\mathcal{T}$ of functions in S, the more general problem of finding sharp estimates for the functional $|a_3 - \mu a_2^2|$ ($\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$) is popularly known as the Fekete–Szegö problem for $\mathcal{T}$. Ali et al. [1] obtained sharp bounds for the Fekete–Szegö functional denoted by $|b_{2k+1} - \mu b_{k+1}^2|$ associated with the k^{th} root transform $\left[f(z^k)\right]^{1/k}$ of the function given in (1.1), belonging to certain subclasses of S. We refer to $|H_2(2)|^{1/k}$ as the second Hankel determinant for the k^{th} root transform associated with the function f. For our discussion in this paper, we consider the Hankel determinant given by $|H_2(2)|^{1/k}$. Motivated by the results obtained by Ali et al. [1], we obtain sharp upper bound to the functional $|b_{k+1}b_{3k+1} - b_{2k+1}^2|$ for the k^{th} root transform of the function f when it belongs to certain subclass denoted by $Q(\alpha, \beta, \gamma)$ of S, defined as follows.

Definition 1.1. A function $f \in A$ is said to be in the class $Q(\alpha, \beta, \gamma)$ with α, $\beta > 0$ and $0 \le \gamma < \alpha + \beta \le 1$, if it satisfies the condition

$$\operatorname{Re}\left\{\alpha \frac{f(z)}{z} + \beta f'(z)\right\} \ge \gamma, \quad z \in E.$$

This class was considered and studied by Wang et al. [12].

2. Preliminary results

Let $\mathscr{P}$ denote the class of functions

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\operatorname{Re} p(z) > 0$ for any $z \in E$. Here $p(z)$ is called the Carathéodory function [3].

Lemma 2.1 (see [9, 10])**.** *If $p \in \mathscr{P}$, then $|c_k| \leq 2$ for each $k \geq 1$, the inequality is sharp for the function $p_0(z) = (1+z)/(1-z)$.*

Lemma 2.2 (see [4])**.** *The power series for $p(z)$ given in* (2.1) *converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n \in \mathbb{N}, \ c_{-k} = \bar{c}_k,$$

are all non-negative. They are strictly positive except $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$ with $\sum_{k=1}^{m} \rho_k = 1$, t_k real, and $t_k \neq t_j$ for $k \neq j$. In this case, $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [4] is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2 for $n = 2$ and $n = 3$, we have, respectively,

$$2c_2 = c_1^2 + y(4 - c_1^2) \tag{2.2}$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta \tag{2.3}$$

for some complex valued y with $|y| \leq 1$ and for some complex valued ζ with $|\zeta| \leq 1$. To obtain our result, we refer to the classical method initiated by Libera and Złotkiewicz [6], which has been used widely.

3. Main result

Theorem 3.1. *If f given by* (1.1) *belongs to $Q(\alpha, \beta, \gamma)$ with $\alpha, \beta > 0$ and $0 \leq \gamma < \alpha + \beta \leq 1$), then*

$$\left| b_{k+1} b_{3k+1} - b_{2k+1}^2 \right| \leq \left[\frac{2(\alpha + \beta - \gamma)}{k(\alpha + 3\beta)} \right]^2$$

and the inequality is sharp.

Proof. Let $f \in Q(\alpha, \beta, \gamma)$. By virtue of Definition 1.1, there exists an analytic function $p \in \mathscr{P}$ in the open unit disc E with $p(0) = 1$ and $\operatorname{Re} p(z) > 0$ such that

$$\frac{\alpha f(z) + \beta z f'(z) - \gamma z}{(\alpha + \beta - \gamma) z} = p(z). \tag{3.1}$$

Replacing $f(z)$, $f'(z)$ and $p(z)$ with their equivalent series expressions in the relation (3.1), we have

$$\alpha\left\{z+\sum_{n=2}^{\infty}a_nz^n\right\}+\beta z\left\{1+\sum_{n=2}^{\infty}na_nz^{n-1}\right\}-\gamma z$$
$$=(\alpha+\beta-\gamma)z\left\{1+\sum_{n=1}^{\infty}c_nz^n\right\}.$$

Upon simplification, we obtain

$$\begin{aligned}(\alpha+2\beta)a_2+(\alpha+3\beta)a_3z+(\alpha+4\beta)a_4z^2+\dots\\ =(\alpha+\beta-\gamma)\left(c_1+c_2z+c_3z^2+\dots\right).\end{aligned}\tag{3.2}$$

Equating the coefficients of like powers of z^0, z^1 and z^2, respectively, on both sides of (3.2), we get

$$a_2=\frac{\alpha+\beta-\gamma}{\alpha+2\beta}c_1,\quad a_3=\frac{\alpha+\beta-\gamma}{\alpha+3\beta}c_2,\quad a_4=\frac{\alpha+\beta-\gamma}{\alpha+4\beta}c_3.\tag{3.3}$$

For a function f given by (1.1), a computation shows that

$$\begin{aligned}\left[f(z^k)\right]^{\frac{1}{k}}&=\left[z^k+\sum_{n=2}^{\infty}a_nz^{nk}\right]^{\frac{1}{k}}\\ &=z+\frac{1}{k}a_2z^{k+1}+\left\{\frac{1}{k}a_3+\frac{1-k}{2k^2}a_2^2\right\}z^{2k+1}\\ &\quad+\left\{\frac{1}{k}a_4+\frac{1-k}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{6k^3}a_2^3\right\}z^{3k+1}+\dots.\end{aligned}\tag{3.4}$$

The expressions (1.2) and (3.4) yield

$$\begin{aligned}b_{k+1}&=\frac{1}{k}a_2,\quad b_{2k+1}=\frac{1}{k}a_3+\frac{1-k}{2k^2}a_2^2,\\ b_{3k+1}&=\frac{1}{k}a_4+\frac{1-k}{k^2}a_2a_3+\frac{(1-k)(1-2k)}{6k^3}a_2^3.\end{aligned}\tag{3.5}$$

Simplifying the relations (3.3) and (3.5), we get

$$\begin{aligned}b_{k+1}&=\frac{\alpha+\beta-\gamma}{k(\alpha+2\beta)}c_1,\\ b_{2k+1}&=\frac{\alpha+\beta-\gamma}{k}\left[\frac{1}{(\alpha+3\beta)}c_2+\frac{(1-k)(\alpha+\beta-\gamma)}{2k(\alpha+2\beta)^2}c_1^2\right],\\ b_{3k+1}&=\frac{\alpha+\beta-\gamma}{k}\left[\frac{1}{(\alpha+4\beta)}c_3+\frac{(1-k)(\alpha+\beta-\gamma)}{k(\alpha+2\beta)(\alpha+3\beta)}c_1c_2\right.\\ &\quad\left.+\frac{(1-k)(1-2k)(\alpha+\beta-\gamma)^2}{6k^2(\alpha+2\beta)^3}c_1^3\right].\end{aligned}\tag{3.6}$$

Substituting the values of b_{k+1}, b_{2k+1} and b_{3k+1} from (3.6) in the second Hankel determinant to the k^{th} transform for the function $f \in Q(\alpha,\beta,\gamma)$, which simplifies to give

$$\begin{aligned}|b_{k+1}b_{3k+1} - b_{2k+1}^2| &= \frac{(\alpha+\beta-\gamma)^2}{12k^4(\alpha+2\beta)^4(\alpha+3\beta)^2(\alpha+4\beta)} \\ &\times \Big|12k^2(\alpha+\beta)^3(\alpha+3\beta)^2c_1c_3 - 12k^2(\alpha+2\beta)^4(\alpha+4\beta)c_2^2 \\ &+ (k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)c_1^4\Big|.\end{aligned} \tag{3.7}$$

The above expression is equivalent to

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| = t\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right|, \tag{3.8}$$

where

$$t = \frac{(\alpha+\beta-\gamma)^2}{12k^4(\alpha+2\beta)^4(\alpha+3\beta)^2(\alpha+4\beta)} \tag{3.9}$$

and

$$\begin{aligned} d_1 &= 12k^2(\alpha+2\beta)^3(\alpha+3\beta)^2, \\ d_2 &= 12k^2(\alpha+2\beta)^4(\alpha+4\beta), \\ d_3 &= (k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta). \end{aligned} \tag{3.10}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.3), respectively, from Lemma 2.2 on the right-hand side of (3.8), we have

$$\begin{aligned}\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| &= \Big|\frac{1}{4}d_1c_1\{c_1^3 + 2c_1(4-c_1^2)y - c_1(4-c_1^2)y^2 \\ &+ 2(4-c_1^2)(1-|y|^2)\zeta\} + \frac{1}{4}d_2\{c_1^2 + y(4-c_1^2)\}^2 + d_3c_1^4\Big|.\end{aligned}$$

Using the triangle inequality and the fact that $|\zeta| < 1$, after simplifying we get

$$\begin{aligned}4\left|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4\right| \le \Big|&(d_1+d_2+4d_3)c_1^4 \\ &+2d_1c_1(4-c_1^2) + 2(d_1+d_2)c_1^2(4-c_1^2)|y| \\ &- \left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\}(4-c_1^2)|y|^2\Big|.\end{aligned} \tag{3.11}$$

Using the values of d_1, d_2 and d_3 from (3.10), we can write

$$\begin{aligned}(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2 &= 12k^2(\alpha+2\beta)^3 \\ &\times \left\{\beta^2c_1^2 + 2(\alpha+3\beta)^2c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\}.\end{aligned} \tag{3.12}$$

Consider

$$\beta^2c_1^2 + 2(\alpha+3\beta)^2c_1 + 4(\alpha+2\beta)(\alpha+4\beta)$$

$$= \beta^2\left[\left\{c_1 + \frac{(\alpha+3\beta)^2}{\beta^2}\right\}^2 - \left\{\sqrt{\frac{\alpha^4 + 49\beta^4 + 50\alpha^2\beta^2 + 84\alpha\beta^3 + 12\alpha^3\beta}{\beta^4}}\right\}^2\right]$$

$$= \beta^2 \left[c_1 + \left\{ \frac{(\alpha+3\beta)^2}{\beta^2} + \sqrt{\frac{\alpha^4 + 49\beta^4 + 50\alpha^2\beta^2 + 84\alpha\beta^3 + 12\alpha^3\beta}{\beta^4}} \right\} \right]$$
$$\times \left[c_1 + \left\{ \frac{(\alpha+3\beta)^2}{\beta^2} - \sqrt{\frac{\alpha^4 + 49\beta^4 + 50\alpha^2\beta^2 + 84\alpha\beta^3 + 12\alpha^3\beta}{\beta^4}} \right\} \right].$$

Since $c_1 \in [0,2]$, noting that $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a, b \geq 0$ on the right-hand side of above expression, we have

$$\begin{aligned} &\beta^2 c_1^2 + 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta) \\ &\qquad \geq \beta^2 c_1^2 - 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta). \end{aligned} \tag{3.13}$$

From the relations (3.12) and (3.13), we get

$$\begin{aligned} &-\left\{(d_1+d_2)c_1^2 + 2d_1c_1 - 4d_2\right\} \leq -12k^2(\alpha+2\beta)^3 \\ &\qquad \times \left\{\beta^2 c_1^2 - 2(\alpha+3\beta)^2 c_1 + 4(\alpha+2\beta)(\alpha+4\beta)\right\}. \end{aligned} \tag{3.14}$$

Substituting the calculated values from (3.10) and (3.14) on the right-hand side of (3.11), we have

$$\begin{aligned} 4|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4| \leq \Big| &\big[12k^2(\alpha+2\beta)^3\beta^2 \\ &- 4(k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)\big]c_1^4 \\ &+ 24k^2(\alpha+2\beta)^3\left\{(\alpha+3\beta)^2c_1 + \beta^2c_1^2|y|\right\}(4-c_1^2) \\ &- 12k^2(\alpha+2\beta)^3\left\{\beta^2c_1^2 - 2(\alpha+3\beta)^2c_1\right. \\ &\left.+4(\alpha+2\beta)(\alpha+4\beta)\right\}(4-c_1^2)|y|^2\Big|. \end{aligned}$$

Choosing $c_1 = c \in [0,2]$, applying the triangle inequality and replacing $|y|$ by μ on the right-hand side of the above inequality, we obtain

$$4|d_1c_1c_3 + d_2c_2^2 + d_3c_1^4| \leq F(c,\mu), \tag{3.15}$$

where

$$\begin{aligned} F(c,\mu) = &\left\{12k^2(\alpha+2\beta)^3\beta^2\right. \\ &\left.- 4(k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)\right\}c^4 \\ &+ 24k^2(\alpha+2\beta)^3\left\{(\alpha+3\beta)^2c + \beta^2c^2\mu\right\}(4-c^2) \\ &+ 12k^2(\alpha+2\beta)^3\left\{\beta^2c^2 - 2(\alpha+3\beta)^2c\right. \\ &\left.+4(\alpha+2\beta)(\alpha+4\beta)\right\}(4-c^2)\mu^2. \end{aligned} \tag{3.16}$$

Next, we maximize the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ in (3.16) partially with respect to μ, we get

$$\begin{aligned}\frac{\partial F}{\partial \mu} &= 24k^2(\alpha+2\beta)^3\left[\beta^2c^2\right.\\ &\quad\left.+\left\{\beta^2c^2-2(\alpha+3\beta)^2c+4(\alpha+2\beta)(\alpha+4\beta)\right\}\mu\right](4-c^2).\end{aligned}\tag{3.17}$$

For $0<\mu<1$, for fixed c with $0<c<2$ and $\alpha,\beta>0$, from (3.17) we observe that $\frac{\partial F}{\partial \mu}>0$. Consequently, $F(c,\mu)$ becomes an increasing function of μ and, hence, $F(c,\mu)$ cannot have a maximum value at any point in the interior of the closed region $[0,2]\times[0,1]$. Further, for fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le1} F(c,\mu)=F(c,1)=G(c).\tag{3.18}$$

Simplifying the relations (3.16) and (3.18), we obtain

$$\begin{aligned}G(c) &= -4\big\{(k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)\\ &\quad+6k^2\beta^2(\alpha+2\beta)^3\big\}c^4-48k^2(\alpha+2\beta)^3(\alpha^2+6\alpha\beta+6\beta^2)c^2\\ &\quad+192k^2(\alpha+2\beta)^4(\alpha+4\beta),\end{aligned}\tag{3.19}$$

and, consequently,

$$\begin{aligned}G'(c) &= -16\big\{(k^2-1)(\alpha+\beta-\gamma)^2(\alpha+3\beta)^2(\alpha+4\beta)\\ &\quad+6k^2\beta^2(\alpha+2\beta)^3\big\}c^3-96k^2(\alpha+2\beta)^3(\alpha^2+6\alpha\beta+6\beta^2)c.\end{aligned}\tag{3.20}$$

From the expression (3.20), we observe that $G'(c)\le 0$ for all values of $c\in[0,2]$ and for fixed values of α, $\beta>0$, where $0\le\gamma<\alpha+\beta\le 1$. Therefore, $G(c)$ becomes a monotonically decreasing function of c in the interval $[0,2]$ and hence it attains the maximum value at $c=0$ only. From (3.19), the maximum value of $G(c)$ is given by

$$\max_{0\le c\le 2} G(c)=G(0)=192k^2(\alpha+2\beta)^4(\alpha+4\beta).\tag{3.21}$$

Considering, only the maximum value of $G(c)$ at $c=0$, from the relations (3.15) and (3.21), after simplifying, we get

$$|d_1c_1c_3+d_2c_2^2+d_3c_1^4|\le 48k^2(\alpha+2\beta)^4(\alpha+4\beta).\tag{3.22}$$

Simplifying the expressions (3.8) and (3.22) together with (3.9), we obtain

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|\le\left[\frac{2(\alpha+\beta-\gamma)}{k(\alpha+3\beta)}\right]^2.\tag{3.23}$$

If we set $c_1=c=0$ and select $y=1$ in (2.2) and (2.3), we find that $c_2=2$ and $c_3=0$. Using these values in (3.22), we observe that equality is attained, which shows that our result is sharp. For these values, we derive the extremal function from (2.1), given by

$$\alpha\frac{f(z)}{z}+\beta f'(z)-\gamma=\frac{\alpha f(z)+\beta zf'(z)-\gamma z}{(\alpha+\beta-\gamma)z}=1+2z^2+2z^4-\cdots=\frac{1-z^2}{1+z^2}.$$

This completes the proof of our theorem. □

Remark 3.2. For the choice of $\alpha=(1-\sigma)$, $\beta=\sigma$ and $\gamma=0$, we get

$$(\alpha,\beta,\gamma)=((1-\sigma),\sigma,0),$$

for which, from (3.23), upon simplification, we obtain

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|\le\frac{4}{(1+2\sigma)^2},\quad 0\le\sigma\le1.$$

This result is a special case of that of Murugusundaramoorthy and Magesh [7].

Remark 3.3. Selecting $k=1$, $\alpha=0$, $\beta=1$ and $\gamma=0$ in (3.23), we obtain

$$|b_2b_4-b_3^2|\le\frac{4}{9}.$$

This result coincides with that of Janteng et al. [5].

Remark 3.4. Choosing $k=1$ in (3.23), we obtain

$$|b_2b_4-b_3^2|\le\frac{4(\alpha+\beta-\gamma)^2}{(\alpha+3\beta)^2}.$$

This result coincides with that of Vamshee Krishna and RamReddy [11].

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References

[1] R. M. Ali, S. K. Lee, V. Ravichandran, and S. Supramaniam, *The Fekete-Szegö coefficient functional for transforms of analytic functions*, Bull. Iranian Math. Soc. **25** (2009), 119–142.

[2] L. de Branges, *A proof of Bieberbach conjecture*, Acta Math. **154** (1985), 137–152.

[3] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften **259**, Springer, New York, 1983.

[4] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, Chelsea Publishing Co., New York, 1984.

[5] A. Janteng, S. A. Halim, and M. Darus, *Coefficient inequality for a function whose derivative has a positive real part*, JIPAM J. Inequal. Pure Appl. Math. **7** (2006), Art. 50, 5 pp.

[6] R. J. Libera and E. J. Złotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathscr{P}$*, Proc. Amer. Math. Soc. **87** (1983), 251–257.

[7] G. Murugusundaramoorthy and N. Magesh, *Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant*, Bull. Math. Anal. Appl. **1** (2009), 85–89.
[8] Ch. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc. **41** (1966), 111–122.
[9] Ch. Pommerenke, *Univalent Functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
[10] B. Simon, *Orthogonal Polynomials on the Unit Circle. Part 1. Classical Theory*, American Mathematical Society Colloquium Publications **54**, American Mathematical Society, Providence, RI, 2005.
[11] D. Vamshee Krishna and T. RamReddy, *Coeffiient inequality for certain subclass of analytic functions*, Armen. J. Math. 4 (2012), 98–105.
[12] Z.-G. Wang, C.-Y. Gao, and S.-M. Yuan, *On the univalency of certain analytic functions*, JIPAM J. Inequal. Pure Appl. Math. **7** (2006), Art. 9, 4 pp.

Department of Mathematics, Kakatiya University, Warangal 506 009, T.S., India
E-mail address: reddytr2@gmail.com

Sri Venkateswara College of Engineering and Technology, Etcherla 532 410, A.P., India
E-mail address: shaliniraj1005@gmail.com

Department of Mathematics, GIT, GITAM University, Visakhapatnam 530 045, A.P., India
E-mail address: vamsheekrishna1972@gmail.com
E-mail address: bvlmaths@gmail.com