Certain Diophantine equations involving balancing and Lucas-balancing numbers

Prasanta Kumar Ray

Abstract. It is well known that if \( x \) is a balancing number, then the positive square root of \( 8x^2 + 1 \) is a Lucas-balancing number. Thus, the totality of balancing number \( x \) and Lucas-balancing number \( y \) are seen to be the positive integral solutions of the Diophantine equation \( 8x^2 + 1 = y^2 \). In this article, we consider some Diophantine equations involving balancing and Lucas-balancing numbers and study their solutions.

1. Introduction

The concept of balancing numbers came into existence after the paper [2] by Behera and Panda wherein, they defined a balancing number \( n \) as a solution of the Diophantine equation \( 1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r) \), calling \( r \) the balancers corresponding to \( n \). They also proved that, \( x \) is a balancing number if and only if \( 8x^2 + 1 \) is a perfect square. In a subsequent paper [7], Panda studied several fascinating properties of balancing numbers calling the positive square root of \( 8x^2 + 1 \), a Lucas-balancing number for each balancing number \( x \). In [7], Panda observed that the Lucas-balancing numbers are associated with balancing numbers in the way Lucas numbers are attached to Fibonacci numbers. Thus, all balancing numbers \( x \) and corresponding Lucas-balancing numbers \( y \) are positive integer solutions of the Diophantine equation \( 8x^2 + 1 = y^2 \). Though the relationship between balancing and Lucas-balancing numbers is non-linear, like Fibonacci and Lucas numbers, they share the same linear recurrence \( x_{n+1} = 6x_n - x_{n-1} \), while initial values of balancing numbers are \( x_0 = 0 \), \( x_1 = 1 \) and for Lucas-balancing numbers \( x_0 = 1 \), \( x_1 = 3 \). Demirtürk and Keskin [3] studied certain Diophantine equations relating to Fibonacci and Lucas numbers. Recently,
Keskin and Karaatli [4] have developed some interesting properties for balancing numbers and square triangular numbers. Alvarado et al. [1], Liptai [5, 6], and Szalay [20] studied certain Diophantine equations relating to balancing numbers. The objective of this paper is to study some Diophantine equations involving balancing and Lucas-balancing numbers. The solutions are obtained in terms of these numbers.

2. Preliminaries

In this section, we present some definitions and identities on balancing and Lucas-balancing numbers which we need in the sequel. As usual, we denote the $n$th balancing and Lucas-balancing numbers by $B_n$ and $C_n$, respectively. It is well known from [7] that $C_n = \sqrt{8}B_n^2 + 1$. The sequences \{\(B_n\)\} and \{\(C_n\)\} satisfy the recurrence relations

\[B_{n+1} = 6B_n - B_{n-1}, \quad B_0 = 0, \quad B_1 = 1; \quad C_{n+1} = 6C_n - C_{n-1}, \quad C_0 = 1, \quad C_1 = 3.\]

The balancing numbers and Lucas balancing numbers can also be defined for negative indices by modifying their recurrences as

\[B_{n-1} = 6B_n - B_{n+1}; \quad C_{n-1} = 6C_n - C_{n+1},\]

and calculating backwards. It is easy to see that $B_{-1} = -1$. Because $B_0 = 0$, we can check easily that all negatively subscripted balancing numbers are negative and that $B_{-n} = -B_n$. By a similar argument, it is easy to verify that $C_{-n} = C_n$. Binet’s formulas for balancing and Lucas-balancing numbers are, respectively,

\[B_n = \frac{\alpha_1^{2n} - \alpha_2^{2n}}{4\sqrt{2}} \text{ and } C_n = \frac{\alpha_1^{2n} + \alpha_2^{2n}}{2},\]

where $\alpha_1 = 1 + \sqrt{2}$, $\alpha_2 = 1 - \sqrt{2}$, which are units of the ring $\mathbb{Z}(\sqrt{2})$. The totality of units of $\mathbb{Z}(\sqrt{2})$ is given by

\[U = \{\alpha_1^n, \alpha_2^n, -\alpha_1^n, -\alpha_2^n : n \in \mathbb{Z}\}.\]

The set $U$ can be partitioned into two subsets $U_1$ and $U_2$ such that $U_1 = \{u \in U : u\bar{u} = 1\}$ and $U_2 = \{u \in U : u\bar{u} = -1\}$. Since $\alpha_1 = \alpha_2$ and $\alpha_1\alpha_2 = -1$, it follows that

\[U_1 = \{\alpha_1^{2n}, \alpha_2^{2n}, -\alpha_1^{2n}, -\alpha_2^{2n} : n \in \mathbb{Z}\},\]

\[U_2 = \{\alpha_1^{2n+1}, \alpha_2^{2n+1}, -\alpha_1^{2n+1}, -\alpha_2^{2n+1} : n \in \mathbb{Z}\}.\]

We also write $\lambda_1 = \alpha_1^2 = 3 + \sqrt{8}$, $\lambda_2 = \alpha_2^2 = 3 - \sqrt{8}$ and therefore, we have $\lambda_1\lambda_2 = 1$. Thus, the set $U_1$ can be written as

\[U_1 = \{\lambda_1^n, \lambda_2^n, -\lambda_1^n, -\lambda_2^n : n \in \mathbb{Z}\}.\]
We also need the following identities (see [9]) while establishing certain identities and solving some Diophantine equations in the subsequent section. The first identity is

\[ B_n^2 = B_{n-1}B_{n+1} + 1. \]

Using the recurrence relation \( B_{n+1} = 6B_n - B_{n-1} \), the identity reduces to

\[ B_n^2 - 6B_nB_{n-1} + B_{n-1}^2 = 1, \]

which we may call as Cassini’s formula for balancing numbers. Similar identities for Lucas-balancing numbers are

\[ C_n^2 = C_{n-1}C_{n+1} - 8, \]
\[ C_n^2 - 6C_nC_{n+1} + C_{n-1}^2 = -8. \]

The idea of naming the identity as Cassini’s formula comes from the literature, where this formula for Fibonacci numbers

\[ F_{n-1}F_{n+1} - F_n^2 = (-1)^n \]

or, equivalently, \( F_{n-1}F_{n+1} - F_n^2 = (-1)^{n-1} \)

is available. Some other important identities are found in [8, 9]:

\[ B_{n+1} - B_{n-1} = 2C_n, \]
\[ B_{n+1} - B_{n-1} = 2C_n, \]
\[ B_m + n = B_mC_n + C_mB_n, \]
\[ B_m - n = B_mC_n - C_mB_n, \]
\[ C_{m+n} = C_mC_n + 8B_mB_n, \]
\[ C_{m-n} = C_mC_n - 8B_mB_n, \]
\[ C_{m+n} = B_{m+1}C_n - C_{n-1}B_m. \]

3. Some identities involving balancing and Lucas-balancing numbers

There are several known identities involving balancing, cobalancing and Lucas-balancing numbers. The interested readers are referred to [7]–[19]. In this section, we only present some new identities involving balancing and Lucas-balancing numbers.

The following two theorems are about nonlinear identities on balancing and Lucas-balancing numbers.

**Theorem 3.1.** For any three integers \( k, m \) and \( n \),

\[ C_{m+n}^2 + 16B_{k-n}C_{m+n}B_{m+k} = 8B_{m+k}^2 + C_{k-n}^2. \]

**Proof.** By virtue of the identities \( B_{m\pm n} = B_mC_n \pm C_mB_n \) and \( C_{m\pm n} = C_mC_n \pm 8B_mB_n \), we obtain

\[ \begin{bmatrix} C_n & 8B_n \\ B_k & C_k \end{bmatrix} \begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix}. \]
Since \( \begin{bmatrix} C_n & 8B_n \\ B_k & C_k \end{bmatrix} = C_{n-k} \) which never vanishes, we have
\[
\begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_n & 8B_n \\ B_k & C_k \end{bmatrix}^{-1} \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix} = \frac{1}{C_{n-k}} \begin{bmatrix} C_k & -8B_n \\ -B_k & C_n \end{bmatrix} \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix}.
\]
This implies that
\[
C_m = \frac{C_kC_{m+n} - 8B_nB_{m+k}}{C_{n-k}} \quad \text{and} \quad B_m = \frac{C_nB_{m+k} - 8B_kC_{m+n}}{C_{n-k}}.
\]
Since \( C_m^2 - 8B_m^2 = 1 \), we have
\[
\left( \frac{C_kC_{m+n} - 8B_nB_{m+k}}{C_{n-k}} \right)^2 - 8 \left( \frac{C_nB_{m+k} - 8B_kC_{m+n}}{C_{n-k}} \right)^2 = 1,
\]
from which the identity follows. \( \square \)

**Theorem 3.2.** If \( k, m \) and \( n \) are three integers such that \( k \neq n \), then
\[
C_{m+n}^2 + C_{m+k}^2 + 8B_{k-n}^2 = 2C_{k-n}C_{m+n}C_{m+k}.
\]

**Proof.** By virtue of the identities \( B_{m \pm n} = B_mB_n \pm C_mB_n \) and \( C_{m \pm n} = C_mC_n \pm 8B_mB_n \), we obtain
\[
\begin{bmatrix} C_n & 8B_n \\ C_k & 8B_k \end{bmatrix} \begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix}.
\]
Since \( \begin{bmatrix} C_n & 8B_n \\ C_k & 8B_k \end{bmatrix} = -8B_{n-k} \), and because \( k \neq n \), this determinant is non-vanishing. Therefore, we have
\[
\begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} C_n & 8B_n \\ C_k & 8B_k \end{bmatrix}^{-1} \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix} = \frac{1}{B_{n-k}} \begin{bmatrix} 8B_k & -8B_n \\ -C_k & -C_n \end{bmatrix} \begin{bmatrix} C_{m+n} \\ B_{m+k} \end{bmatrix},
\]
which implies that
\[
C_m = -\frac{B_kC_{m+n} - B_nC_{m+k}}{8B_{n-k}} \quad \text{and} \quad B_m = -\frac{C_nC_{m+k} - C_kC_{m+n}}{8B_{n-k}}.
\]
Since \( C_m^2 - 8B_m^2 = 1 \), we have
\[
\left( \frac{B_kC_{m+n} - B_nC_{m+k}}{8B_{n-k}} \right)^2 - 8 \left( \frac{C_nC_{m+k} - C_kC_{m+n}}{8B_{n-k}} \right)^2 = 1,
\]
and the required identity follows. \( \square \)

Using the matrix multiplication
\[
\begin{bmatrix} B_n & C_n \\ B_k & C_k \end{bmatrix}^{-1} \begin{bmatrix} C_m \\ B_m \end{bmatrix} = \begin{bmatrix} B_{m+n} \\ B_{m+k} \end{bmatrix},
\]
it is easy to prove the following theorem.
Theorem 3.3. If \( k, m \) and \( n \) are three integers such that \( k \neq n \), then
\[
B_{m+n}^2 + B_{m+k}^2 - B_{k-n}^2 = 2C_{k-n}B_{m+n}B_{m+k}.
\]

4. Some Diophantine equations involving balancing and Lucas-balancing numbers

The identities of Section 3 induce the following three Diophantine equations:

\[
x^2 + 16B_n xy - 8y^2 = C_n^2, \quad (4.1)
\]
\[
x^2 - 2C_n xy + y^2 + C_n^2 = 1, \quad (4.2)
\]
\[
x^2 - 2C_n xy + y^2 = B_n^2. \quad (4.3)
\]

Before the study of these equations, we present the Diophantine equation
\[
x^2 - 6xy + y^2 = 1 \quad (4.4)
\]
resulting out of Casini’s formula for balancing numbers. The following theorem shows that all solutions of (4.4) are consecutive pairs of balancing numbers only.

Theorem 4.1. All solutions of the Diophantine equation (4.4) are consecutive pairs of balancing numbers only.

Proof. After factorization, the Diophantine equation (4.4) takes the form
\[
(\lambda_1 x - y)(\lambda_2 x - y) = 1,
\]
where \( \lambda_1 = 3 + \sqrt{8} \), \( \lambda_2 = 3 - \sqrt{8} \). This suggests that \( (\lambda_1 x - y) \) and \( (\lambda_2 x - y) \) are units of \( \mathbb{Z}(\sqrt{2}) \), conjugate to each other, and are members of \( U_1 \). Thus for some integer \( n \), we have the following four cases.

Case 1 : \( (\lambda_1 x - y) = \lambda_1^n \) and \( (\lambda_2 x - y) = \lambda_2^n \),
Case 2 : \( (\lambda_1 x - y) = \lambda_2^n \) and \( (\lambda_2 x - y) = \lambda_1^n \),
Case 3 : \( (\lambda_1 x - y) = -\lambda_1^n \) and \( (\lambda_2 x - y) = -\lambda_2^n \),
Case 4 : \( (\lambda_1 x - y) = -\lambda_2^n \) and \( (\lambda_2 x - y) = -\lambda_1^n \).

Solving the equation for Case 1, using Cramer’s rule, we find
\[
\begin{vmatrix}
\lambda_1^n & -1 \\
\lambda_2^n & -1 \\
\lambda_1 & -1 \\
\lambda_2 & -1
\end{vmatrix}
= \lambda_1^n - \lambda_2^n = B_n,
\]
\[
y = \frac{1}{\lambda_1 - \lambda_2} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = B_{n-1}.
\]
The solution in Case 2 is
\[
x = \begin{vmatrix} \lambda_2^n & -1 \\ \lambda_1^n & -1 \end{vmatrix} = -B_n = B_{-n}, \quad y = \begin{vmatrix} \lambda_1 & \lambda_2^n \\ \lambda_1 & -1 \end{vmatrix} = -B_{n+1} = B_{-(n+1)}.
\]

Finally, the solutions in Case 3 and Case 4 are, respectively,
\[
x = B_{-n}, \quad y = B_{-(n-1)} \quad \text{and} \quad x = B_n, \quad y = B_{n+1}.
\]

Thus, in all cases, the solutions of the Diophantine equation (4.4) are consecutive pairs of balancing numbers only. □

Using Theorem 4.1, we can find solutions of a Diophantine equation derived from the identity of Theorem 3.1.

**Theorem 4.2.** All solutions of the Diophantine equation (4.1) are
\[
(x, y) = (C_{m-n}, B_m), (-C_{m+n}, B_m), (-C_{m-n}, -B_m), (C_{m+n}, -B_m) \quad (4.5)
\]
for \(m, n \in \mathbb{Z}\).

**Proof.** The equation (4.1) can be rewritten as
\[
(x + 8B_n y)^2 = C_n^2 (8y^2 + 1),
\]
implying that \(8y^2 + 1\) is a perfect square. Hence, \(y\) is a balancing number. So, we may take \(y = \pm B_m\) (\(y = -B_m\) is equivalent to \(y = B_{-m}\)), \(8y^2 + 1 = C_n^2\).

When \(y = B_m\), we have \(x = -8B_mB_n \pm C_mC_n\) or \(x = 8B_mB_n \pm C_mC_n\) and therefore, \(x = \pm C_mC_n - 8B_mB_n\), i.e. \(x = C_{m-n}\) or \(-C_{m+n}\). When \(y = -B_m\), we have \(x = -C_{m-n}\) or \(C_{m+n}\). Thus, the totality of solutions of (4.1) is given by (4.5) for \(m, n \in \mathbb{Z}\). □

**Theorem 4.3.** All solutions of the Diophantine equation (4.2) are given by
\[
(x, y) = (-C_{m-n}, C_m), (-C_{m+n}, C_m), (C_{m+n}, -C_m), (C_{m-n}, -C_m) \quad (4.6)
\]
for \(m, n \in \mathbb{Z}\).

**Proof.** The equation (4.2) can be rewritten as
\[
(x + C_n y)^2 = (C_n^2 - 1)(y^2 - 1) = 8B_n^2 (y^2 - 1),
\]
implying that \(8(y^2 - 1)\) is a perfect square. Thus, \(y\) is odd and hence \(\frac{y^2 - 1}{8}\) is a perfect square. Since \(\frac{y-1}{2}\) and \(\frac{y+1}{2}\) are consecutive integers, it follows that \(\frac{y^2 - 1}{8}\) is a square triangular number. That is, it is the square of a balancing number, say
\[
\frac{y^2 - 1}{8} = B_m^2
\]
and hence \( y^2 = 8B_n^2 + 1 \) for some \( m \), so that \( y = \pm C_m \). Consequently, 
\[(x + C_ny)^2 = 8B_n^2(y^2 - 1)\]
is equivalent to \( x + C_mC_n = \pm 8B_mB_n \) if \( y = C_m \), and \( x - C_mC_n = \pm 8B_mB_n \) if \( y = -C_m \). Thus, the totality of solutions of (4.2) is given by (4.6) for \( m, n \in \mathbb{Z} \).

In the following theorem, we consider the Diophantine equation (4.3) which may be considered as a generalization of the Diophantine equation discussed in Theorem 4.1.

**Theorem 4.4.** All solutions of the Diophantine equation (4.3) are given by
\[
(x, y) = (B_{m-n}, B_m), (-B_{m+n}, -B_m), (-B_{m-n}, B_m), (C_{m+n}, -B_m)
\]
for \( m, n \in \mathbb{Z} \).

**Proof.** The equation (4.3) can be rewritten as
\[
(x + C_ny)^2 = (C_n^2 - 1)(y^2 - 1) = B_n^2(8y^2 + 1),
\]
which suggests that \( 8y^2 + 1 \) is a perfect square. Thus, \( y = \pm B_m \) (as usual \( y = -B_m \) is equivalent to \( y = B_{-m} \)) and hence \( 8y^2 + 1 = C_m^2 \). Now \( x \) can be obtained from \( x + C_y = \pm C_mB_n \) and therefore, the totality of solutions is given by (4.7) for \( m, n \in \mathbb{Z} \).

In the remaining theorems, we present some Diophantine equations where the proofs require certain divisibility properties of balancing and Lucas-balancing numbers.

**Theorem 4.5.** The solutions of the Diophantine equation
\[
x^2 + 2C_nxy + y^2 = 1
\]
are given by
\[
(x, y) = \left( \frac{-B_{(k+1)n}}{B_n}, \frac{B_{nk}}{B_n} \right), \left( \frac{-B_{(k-1)n}}{B_n}, \frac{B_{nk}}{B_n} \right) \quad (m, n \in \mathbb{Z}).
\]

**Proof.** The equation (4.8) can be rewritten as
\[
(x + C_ny)^2 = (C_n^2 - 1)(y^2 - 1) = 8B_n^2y^2 + 1,
\]
which suggests that \( B_ny \) is a balancing number. Letting \( B_ny = B_m \), we have \( y = \frac{B_m}{B_n} \), and since \( y \) is an integer, it follows that \( B_n \) divides \( B_m \). Hence by Theorem 2.8 in [7] (see also [4]), \( n \) divides \( m \). Thus, \( m = nk \) for some integer \( k \) and \( y = \frac{B_{nk}}{B_n} \). Further,
\[
(x + C_ny)^2 = 8B_n^2y^2 + 1 = 8B_{nk}^2 + 1 = C_{nk}^2,
\]
and hence \( x + C_ny = \pm C_{nk} \). It follows that,
\[
x = -\frac{C_nB_{nk}}{B_n} \pm C_{nk} = \frac{-B_{(k+1)n}}{B_n}, \frac{-B_{(k-1)n}}{B_n}.
\]
Thus, the totality of solutions of (4.8) is given by (4.9). □

The following theorem that resembles Theorem 4.2 deals with a Diophantine equation whose proof requires conditions under which a Lucas-balancing numbers divides balancing and Lucas-balancing numbers.

**Theorem 4.6.** The solutions of the Diophantine equation $x^2 + 16B_nxy - 8y^2 = 1$ are given by

$$(x, y) = \left( \frac{-C_{(k+1)n}}{C_n}, \frac{B_{(2k+1)n}}{C_n} \right), \left( \frac{C_{2kn}}{C_n}, \frac{B_{(2k+1)n}}{B_n} \right) \quad (m, n \in \mathbb{Z}). \quad (4.10)$$

**Proof.** The equation $x^2 + 16B_nxy - 8y^2 = 1$ can be rewritten as

$$(x + 8B_ny)^2 = 8C_n^2y^2 + 1,$$

implying that $8C_n^2y^2 + 1$ is a perfect square and $C_ny$ is a balancing number, say $C_ny = B_m$, hence $C_n$ divides $B_m$. It is easy to see that this is possible if and only if $m$ is an even multiple of $n$, and hence $m = 2kn$ for some integer $k$. Thus, $y = \frac{B_{2kn}}{C_n}$. Further,

$$(x + 8B_ny)^2 = 8B_{2kn}^2 + 1 = C_{2kn}^2,$$

and hence

$$x = -8B_ny \pm C_{2kn} = \frac{-8B_nB_{2kn}}{C_n} \pm C_{2kn}.$$ 

Therefore, the totality of solutions is given by (4.10). □

Lastly, we present a theorem which is a variant of Theorems 3.1 and 4.5.

**Theorem 4.7.** The solutions of the Diophantine equation

$$x^2 - 2C_nxy + y^2 = -8B_n^2$$

are given by

$$(x, y) = (C_{m-n}, C_m), (C_{m+n}, C_m), (-C_{m-n}, -C_m), (-C_{m+n}, -C_m) \quad (4.12)$$

for $m, n \in \mathbb{Z}$.

**Proof.** The equation (4.11) can be rewritten as $(x - C_ny)^2 = 8B_n^2(y^2 - 1)$, which suggests that $8(y^2 - 1)$ is a perfect square. Hence $y$ is odd and

$$8(y^2 - 1) = 64 \cdot \frac{1}{2} \cdot \frac{y - 1}{2} \cdot \frac{y + 1}{2}.$$

Since $\frac{y-1}{2}$ and $\frac{y+1}{2}$ are consecutive integers, it follows that $\frac{1}{2} \cdot \frac{y-1}{2} \cdot \frac{y+1}{2}$ is a square triangular number and hence is equal to the square of a balancing number (see [2]), say,

$$\frac{1}{2} \cdot \frac{y - 1}{2} \cdot \frac{y + 1}{2} = B_m^2$$

for some $m$ and we have $8(y^2 - 1) = 64B_m^2$. Thus, $y^2 = 8B_m^2 + 1 = C_m^2$ implying that $y = \pm C_m$, and the equation $(x - C_ny)^2 = 8B_n^2(y^2 - 1)$ is
reduced to $(x - C_n y)^2 = 64 B_m^2 B_n^2$. Therefore, $x - C_n y = \pm 8 B_m B_n$ and the solutions of the equation (4.11) are given by (4.12) for $m, n \in \mathbb{Z}$. □

References


Veer Surendra Sai University of Technology, Odisha, Burla-768018, India
E-mail address: rayprasanta2008@gmail.com