A note on embedding of semigroup amalgams

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Abstract. We give necessary conditions for the embedding of completely regular semigroup, Clifford semigroup and commutative regular semigroup amalgams.

A semigroup amalgam (cf. [3]) is a list $\mathcal{A} \equiv (U; S_1, S_2; \phi_1, \phi_2)$ comprising three semigroups $U$, $S_1$ and $S_2$, and two monomorphisms $\phi_i : U \rightarrow S_i$, $i \in \{1, 2\}$ (recall that monomorphisms of semigroups are precisely the injective semigroup homomorphisms). We say that $\mathcal{A}$ is embeddable if there exist a semigroup $T$ and monomorphisms $\psi_i : S_i \rightarrow T$ such that

1. $\psi_1 \circ \phi_1 = \psi_2 \circ \phi_2$ and
2. $\psi_1(s_1) = \psi_2(s_2)$, $s_1 \in S_1$, $s_2 \in S_2$ implies that $s_1 = \phi_1(u)$, $s_2 = \phi_2(u)$ for some $u \in U$.

If condition (2) is not necessarily satisfied, we call $\mathcal{A}$ weakly embeddable. In [1], Theorem 2.4, Howie proved that a semigroup amalgam $(U; S_1, S_2; \phi_1, \phi_2)$, in which $S_1$ and $S_2$ are both groups, is embeddable if and only if $U$ is also a group. The main objective of this note is to generalize the necessity part of this theorem to the unions and semilattices of groups. A semigroup $S$ is called completely regular if it is a union of groups. We call $S$ a Clifford semigroup if it is a semilattice of groups.

A semigroup $S$ is called regular (cf. [3], pp. 50–51) if there exists a unary operation $s^{-1} : S \rightarrow S$ given by $s \mapsto s^{-1}$ such that $ss^{-1}s = s$, $s^{-1}ss^{-1} = s^{-1}$. The element $s^{-1}$ is called an inverse of $s$. One can show (see [3], Proposition 4.1.1) that $S$ is completely regular if and only if it is regular and $ss^{-1} = s^{-1}s$ for all $s \in S$. Also, $S$ is a Clifford semigroup if and only if it is completely regular and $(ss^{-1})(tt^{-1}) = (tt^{-1})(ss^{-1})$ for all $s, t \in S$ (refer to Theorem 4.2.1 of [3]). A regular semigroup $S$ is termed an inverse semigroup if $(ss^{-1})(tt^{-1}) = (tt^{-1})(ss^{-1})$ for all $s, t \in S$. Thus Clifford semigroups are

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completely regular inverse semigroups. Moreover, every commutative regular semigroup is a Clifford semigroup.

**Theorem 1.** A semigroup amalgam \((U; S_1, S_2; \phi_1, \phi_2)\), in which \(S_1\) and \(S_2\) are completely regular, is embeddable only if \(U\) is also completely regular.

**Proof.** Consider an amalgam \(A \equiv (U; S_1, S_2; \phi_1, \phi_2)\) in which \(S_1\) and \(S_2\) are both completely regular. Suppose \(A\) is embeddable, say in a semigroup \(T\), and consider an element \(u \in U\). Let us denote \(\phi_1(u) = s_1\) and \(\phi_2(u) = s_2\). Now, because \(S_1\) and \(S_2\) are completely regular, there exist inverses \(s_1^{-1} \in S_1\) and \(s_2^{-1} \in S_2\) of \(s_1\) and \(s_2\) respectively. Let \(\psi_i : S_i \rightarrow T\), \(i \in \{1, 2\}\), be the embedding monomorphisms. First note that \(\psi_1(s_1) = \psi_1(\phi_1(u)) = \psi_2(\phi_2(u)) = \psi_2(s_2)\). We can calculate in \(T\):

\[
\psi_1(s_1^{-1}) = \psi_1(s_1^{-1} s_1 s_1^{-1}) \\
= \psi_1(s_1^{-1}) \psi_1(s_1) \psi_1(s_1^{-1}) = \psi_1(s_1^{-1}) \psi_2(s_2) \psi_1(s_1^{-1}) \\
= \psi_1(s_1^{-1}) \psi_2(s_2 \ s_2^{-1} s_2^{-1}) \psi_1(s_1^{-1}) \\
= \psi_1(s_1^{-1}) \psi_2(s_2) \psi_2(s_2^{-1} s_2^{-1}) \psi_1(s_1^{-1}) \\
= \psi_1(s_1^{-1}) \psi_1(s_1) \psi_2(s_2 \ s_2^{-1} s_2^{-1}) \psi_1(s_1) \psi_1(s_1^{-1}) \\
= \psi_1(s_1^{-1} s_1) \psi_2(s_2^{-1} s_2^{-1}) \psi_1(s_1^{-1} s_1) \\
= \psi_1(s_1) \psi_2(s_2^{-1} s_2^{-1}) \psi_1(s_1^{-1} s_1) = \psi_2(s_2) \psi_2(s_2^{-1} s_2^{-1}) \psi_1(s_1^{-1} s_1) \\
= \psi_2(s_2^{-1} s_2^{-1}) \psi_1(s_1^{-1} s_1) = \psi_2(s_2^{-1} s_2^{-1}) \psi_2(s_2) \\
= \psi_2(s_2^{-1} s_2^{-1}) = \psi_2(s_2^{-1} s_2^{-1}) \\
= \psi_2(s_2^{-1}).
\]

Now, using condition (2) of embeddability, there exists \(u' \in U\) such that \(\phi_1(u') = s_1^{-1}, i \in \{1, 2\}\). Then, because \(s_1 s_1^{-1} s_1 = s_1\) implies \(\phi_1^{-1}(s_1 s_1^{-1} s_1) = \phi_1^{-1}(s_1)\), we have \(u'u' = u\) due to injectivity of \(\phi_1\). Similarly we can conclude that \(u' u u' = u'\) and \(u u' = u' u\). Thus \(U\) is completely regular.

**Corollary 1.** It suffices to show that \(U\) is an inverse semigroup. To this end, observe that the uniqueness of \(u'\) (see Theorem 5.1.1 of [3]) in the previous proof follows from the uniqueness of \(s_1^{-1}\).

**Proof.** The corollary follows by noting that there exists \(u \in U\) such that \(u^{-1} \notin U\).
Corollary 2. A semigroup amalgam \((U; S_1, S_2; \phi_1, \phi_2)\), in which \(S_1\) and \(S_2\) are commutative regular semigroups, is embeddable if and only if \(U\) is regular.

Proof. Because \(S_1\) and \(S_2\) are completely regular, the necessity part follows from the argument employed in the above corollary. The sufficiency part follows from Theorem 3.1 of [2]. \(\square\)

Conclusion 1. Can we generalize Theorem 1 to the class of inverse semigroups? Also, can embedding be replaced by weak embedding in Theorem 1?

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References