Semi-symmetric metric connections on pseudosymmetric Lorentzian $\alpha$-Sasakian manifolds

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Abstract. We consider semi-symmetric metric connections on pseudosymmetric Lorentzian $\alpha$-Sasakian manifolds. We study some properties of Weyl pseudosymmetric and Ricci pseudosymmetric Lorentzian $\alpha$-Sasakian manifolds. We also give an example of a pseudosymmetric Lorentzian $\alpha$-Sasakian manifold with a semi-symmetric metric connection.

1. Introduction

The concept of a pseudosymmetric manifold was introduced by M.C. Chaki and B. Chaki (see [3]) and R. Deszcz (see [8]) in two different ways. Various properties of pseudosymmetric manifolds in various metric structures have been studied in both senses (see [3] – [7]). The two types of pseudosymmetric manifolds are different in their nature. We shall study properties of pseudosymmetric manifolds and Ricci pseudosymmetric manifolds with a semi-symmetric metric connection in the Deszcz sense.

A Riemannian manifold $(M, g)$ of dimension $n$ is called pseudosymmetric if the Riemannian curvature tensor $R$ satisfies, for all vector fields $X,Y,U,V,W$ on $M$, the conditions (see [1])


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where \( L_R \in C^\infty(M) \),
\[
R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X; \nabla_Y]Z,
\]

and \( X \wedge Y \) is an endomorphism which is defined by
\[
(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.
\] (1)

The manifold \( M \) is said to be \textit{pseudosymmetric of constant type} if \( L \) is constant. A Riemannian manifold \((M, g)\) is called \textit{semi-symmetric} if \( R.R = 0 \), where \( R.R \) is the derivative of \( R \) by \( R \).

\textbf{Remark 1.1.} From [2] and [8], we know that, for a \((0, k)\)-tensor field \( T \), the \((0, k + 2)\)-tensor fields \( R.T \) and \( Q(g, T) \) are defined by
\[
(R.T)(X_1, \ldots, X_k; X, Y) = (R(X, Y).T)(X_1, \ldots, X_k)
= -T(R(X, Y)X_1, \ldots, X_k) - \cdots - T(X_1, \ldots, R(X, Y)X_k)
\]

and
\[
Q(g, T)(X_1, \ldots, X_k; X, Y) = -((X \wedge Y).T)(X_1, \ldots, X_k)
= T((X \wedge Y)X_1, \ldots, X_k) + \cdots + T(X_1, \ldots, (X \wedge Y)X_k).
\]

Let \( S \) and \( r \) denote the Ricci tensor and the scalar curvature tensor of \( M \) respectively. The operator \( Q \) and the \((0, 2)\)-tensor \( S^2 \) are defined by
\[
S(X, Y) = g(QX, Y)
\]

and
\[
S^2(X, Y) = S(QX, Y). \] (2)

The Weyl conformal curvature operator \( C \) is defined by
\[
C(X, Y) = R(X, Y) - \frac{1}{n - 2} \left[ X \wedge QY + QX \wedge Y - \frac{r}{n - 1} X \wedge Y \right].
\]

If \( C = 0 \), \( n \geq 3 \), then \( M \) is called \textit{conformally flat}. If the tensors \( R.C \) and \( Q(g, C) \) are linearly dependent, then \( M \) is called \textit{Weyl pseudosymmetric}. This is equivalent to the statement that
\[
\]

holds on the set
\[
U_C = \{ x \in M : C \neq 0 \text{ at } x \},
\]

where \( L_C \) is defined on \( U_C \). If \( R.C = 0 \), then \( M \) is called \textit{Weyl semi-symmetric}. If \( \nabla C = 0 \), then \( M \) is called \textit{conformally symmetric} (see [10], [9]).
2. Preliminaries

A differentiable manifold $M$ of dimension $n$ is said to be a Lorentzian $\alpha$-Sasakian manifold if it admits a $(1,1)$-tensor field $\phi$, a vector field $\xi$, a one-form $\eta$, and Lorentzian metric $g$ which satisfy the conditions

$$\phi^2 = I + \eta \otimes \xi,$$
$$\eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta \circ \phi = 0,$$
$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$
$$g(X, \xi) = \eta(X),$$
$$(\nabla_X \phi)(Y) = \alpha \{g(X, Y)\xi + \eta(Y)X\},$$

for all $X, Y \in \chi(M)$, where $\chi(M)$ is the Lie algebra of smooth vector fields on $M$, $\alpha$ is smooth functions on $M$, and $\nabla$ denotes the covariant differentiation operator of Lorentzian metric $g$ (see [11], [10]).

On a Lorentzian $\alpha$-Sasakian manifold, it can be shown that (see [11], [10])

$$\nabla_X \xi = \alpha \phi X,$$
$$\nabla_Y \eta = \alpha g(\phi X, Y).$$

Moreover, on a Lorentzian $\alpha$-Sasakian manifold the following relations hold (see [10]):

$$\eta(R(X, Y)Z) = \alpha^2 [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],$$
$$R(\xi, X)Y = \alpha^2 [g(X, Y)\xi - \eta(Y)X],$$
$$R(X, Y)\xi = \alpha^2 [\eta(Y)X - \eta(X)Y],$$
$$S(\xi, X) = S(X, \xi) = (n - 1)\alpha^2 \eta(X),$$
$$S(\xi, \xi) = -(n - 1)\alpha^2,$$
$$Q\xi = (n - 1)\alpha^2 \xi.$$

The equalities (4) – (8) will be required in the next section.

3. Semi-symmetric metric connection on a Lorentzian $\alpha$-Sasakian manifold

Let $M$ be a Lorentzian $\alpha$-Sasakian manifold with Levi–Civita connection $\nabla$ and let $X, Y, Z \in \chi(M)$. We define a linear connection $D$ on $M$ by

$$D_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi,$$

where $\eta$ is 1-form and $\phi$ is a tensor field of type $(1,1)$. The connection $D$ is said to be semi-symmetric if $\tilde{T}$, the torsion tensor of the connection $D$, satisfies

$$\tilde{T}(X, Y) = \eta(Y)X - \eta(X)Y,$$
and metric if
\[(D_X g)(Y, Z) = 0.\]  \(\text{(11)}\)

The connection \(D\) is said to be semi-symmetric metric if it satisfies (9), (10) and (11).

We shall show the existence of a semi-symmetric metric connection \(D\) on a Lorentzian \(\alpha\)-Sasakian manifold \(M\).

**Theorem 3.1.** Let \(X, Y, Z\) be vector fields on a Lorentzian \(\alpha\)-Sasakian manifold \(M\). Define a connection \(D\) by
\[2g(D_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y) + g(\eta(Y)X - \eta(X)Y, Z) \quad (12)\]
\[+ g(\eta(X)Z - \eta(Z)X, Y) + g(\eta(Y)Z - \eta(Z)Y, X).\]

Then \(D\) is a semi-symmetric metric connection on \(M\).

**Proof.** It can be verified that \(D\) is a linear connection on \(M\). From (12), we have
\[g(D_X Y, Z) - g(D_Y X, Z) = g([X, Y], Z) + \eta(Y)g(X, Z) - \eta(X)g(Y, Z)\]
or
\[D_X Y - D_Y X - [X, Y] = \eta(Y)X - \eta(X)Y\]
or
\[T(X, Y) = \eta(Y)X - \eta(X)Y.\]
Again, from (12), we get
\[2g(D_X Y, Z) + 2g(D_Z X, Y) = 2Xg(Y, Z)\]
or
\[(D_X g)(Y, Z) = 0.\]
This shows that \(D\) is a semi-symmetric metric connection on \(M\). \(\square\)

4. Curvature tensor and Ricci tensor of semi-symmetric metric connection \(D\) on a Lorentzian \(\alpha\)-Sasakian manifold

Let \(\bar{R}(X, Y)Z\) and \(R(X, Y)Z\) be the curvature tensors on a Lorentzian \(\alpha\)-Sasakian manifold \(M\) of a semi-symmetric metric connection \(D\) and of the Riemannian connection \(\nabla\), respectively. A relation between \(\bar{R}(X, Y)Z\) and \(R(X, Y)Z\) is given by
\[
\bar{R}(X, Y)Z = R(X, Y)Z + \alpha [g(\phi X, Z)Y - g(\phi Y, Z)X + g(X, Z)\phi Y - g(Y, Z)\phi X] + \eta(Z) [\eta(Y)X - \eta(X)Y]\]
\[+ g(Y, Z)\phi^2 X - g(X, Z)\phi^2 Y.\]  \(\text{(13)}\)

From (13) we obtain
\[
\bar{S}(X, Y) = S(X, Y) + (n - 2) [g(\phi X, \phi Y) - \alpha g(\phi X, Y)],\]  \(\text{(14)}\)
where $\bar{S}$ and $S$ are the Ricci tensors of the connections $D$ and $\nabla$, respectively. Again
\[
\bar{S}^2(X,Y) = S^2(X,Y) + (n-2) \left[ \{S(\phi X, \phi Y) + S(\phi^2 X, Y)\}
- 2\alpha S(\phi X, Y) \right] + (n-2)^2 \left[ (\alpha^2 + 1)g(\phi X, \phi Y) \right.
- 2\alpha g(\phi X, Y) \bigg].
\] (15)
Contracting (15), we get
\[
\bar{r} = r + (n-1)(n-2),
\] (16)
where $\bar{r}$ and $r$ are the scalar curvatures of the connections $D$ and $\nabla$, respectively.

Let $\bar{C}$ be the conformal curvature tensor of the connection $D$. Then
\[
\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-2} [\bar{S}(Y,Z)X - g(X,Z)\bar{Q}Y + g(Y,Z)\bar{Q}X - \bar{S}(X,Z)Y] + \frac{(n-1)(n-2)}{2} [g(Y,Z)X - g(X,Z)Y],
\] (17)
where $\bar{Q}$ is the Ricci operator of the connection $D$ on $M$ and
\[
\bar{S}(X,Y) = \bar{S}(\bar{Q}X,Y),
\] (18)
\[
\bar{S}^2(X,Y) = \bar{S}(\bar{Q}X,Y).
\] (19)

Now we shall prove the following theorem.

**Theorem 4.1.** Let $M$ be a Lorentzian $\alpha$-Sasakian manifold with a semi-symmetric metric connection $D$. Then the following relations hold:
\[
\bar{R}(\xi,X)Y = \alpha^2 \left[ g(X,Y)\xi - \eta(Y)X \right] = \alpha \left[ \eta(Y)\phi X - g(\phi X, Y)\xi \right],
\] (20)
\[
\eta(\bar{R}(X,Y)Z) = \alpha^2 \left[ g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \right] + \alpha \left[ g(\phi X, Z)\eta(Y) - g(\phi Y, Z)\eta(X) \right],
\] (21)
\[
\bar{R}(X,Y)\xi = \alpha^2 \left[ \eta(Y)X - \eta(X)Y \right] - \alpha \left[ \eta(Y)\phi X - \eta(X)\phi Y \right],
\] (22)
\[
\bar{S}(X,\xi) = \bar{S}(\xi, X) = (n-1)\alpha^2 \eta(X),
\] (23)
\[
\bar{S}^2(X,\xi) = \bar{S}^2(\xi, X) = \alpha^4 (n-1)^2 \eta(X),
\] (24)
\[
\bar{S}(\xi,\xi) = -(n-1)\alpha^2,
\] (25)
\[
\bar{Q}X = QX + (n-2) \left[ \phi^2 X - \alpha \phi X \right],
\] (26)
\[
\bar{Q}\xi = (n-1)\alpha^2 \xi.
\] (27)

**Proof.** Since $M$ is a Lorentzian $\alpha$-Sasakian manifold and $D$ is a semi-symmetric metric connection, replacing $X = \xi$ in (13) and using (3) and (5), we get (20). Using (3) and (4), from (13), we get (21). To prove (22), we put $Z = \xi$ in (13) and then we use (6). Replacing $Y = \xi$ in (14) and using
(7), we get (23). Putting $Y = \xi$ in (15) and using (2) and (7), we get (24). Again, putting $X = Y = \xi$ in (14) and using (8), we get (25). Using (18) and (23), we get (26). Now, putting $X = \xi$ in (26), we get (27). □

5. Lorentzian $\alpha$-Sasakian manifold with a semi-symmetric metric connection $D$ satisfying the condition $\bar{C}\bar{S} = 0$

In this section we shall find out characterization of Lorentzian $\alpha$-Sasakian manifold with a semi-symmetric metric connection $D$ satisfying the condition $\bar{C}\bar{S} = 0$, where

\[
\bar{C}(X,Y)\bar{S}(Z,W) = -\bar{S}(\bar{C}(X,Y)Z,W) - \bar{S}(Z,\bar{C}(X,Y)W)
\]

with $X, Y, Z, W \in \chi(M)$.

**Theorem 5.1.** Let $M$ be an $n$-dimensional Lorentzian $\alpha$-Sasakian manifold with a semi-symmetric metric connection $D$. If $\bar{C}\bar{S} = 0$, then

\[
\frac{1}{n-2}\bar{S}^2(X,Y) = \left[ \alpha^2 + 1 + \frac{r}{(n-1)(n-2)} \right] \left[ \bar{S}(\phi X,Y) - \alpha^2(n-1)g(\phi X,Y) \right]
\]

\[
- \frac{\alpha^2(n-1)g(\phi X,Y)}{n-2} + \frac{\alpha^4(n-1)^2}{n-2}g(X,Y).
\]

**Proof.** From (28), we get

\[
\bar{S}(\bar{C}(X,Y)Z,W) + \bar{S}(Z,\bar{C}(X,Y)W) = 0,
\]

where $X, Y, Z, W \in \chi(M)$. Now, putting $X = \xi$ in (29), we get

\[
\bar{S}(\bar{C}(\xi,X)Y,Z) + \bar{S}(Y,\bar{C}(\xi,X)Z) = 0.
\]

Using (17), (19), (20) and (23), we have that

\[
\bar{S}(\bar{C}(\xi,X)Y,Z) = \left[ \alpha^2 - \frac{(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)} \right]
\]

\[
\times \left[ (n-1)\alpha^2\eta(Z)g(X,Y) - \eta(Y)\bar{S}(X,Z) \right]
\]

\[- \alpha \left[ (n-1)\alpha^2\eta(Z)g(\phi X,Y) - \eta(Y)\bar{S}(\phi X,Z) \right]
\]

\[- \frac{1}{n-2} \left[ (n-1)\alpha^2\eta(Z)\bar{S}(X,Y) - \bar{S}^2(X,Z)\eta(Y) \right],
\]

and

\[
\bar{S}(Y,\bar{C}(\xi,X)Z) = \left[ \alpha^2 - \frac{(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)} \right]
\]

\[
\times \left[ (n-1)\alpha^2\eta(Y)g(X,Z) - \eta(Z)\bar{S}(X,Y) \right]
\]

\[- \alpha \left[ (n-1)\alpha^2\eta(Y)g(\phi X,Z) - \eta(Z)\bar{S}(\phi X,Y) \right]
\]

\[- \frac{1}{n-2} \left[ (n-1)\alpha^2\eta(Y)\bar{S}(X,Z) - \bar{S}^2(X,Y)\eta(Z) \right].
\]
Using (31) and (32) in (30), we get
\[
\alpha^2 - \frac{(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)} \left[ (n-1)\alpha^2 \{\eta(Z)g(X,Y) + \eta(Y)g(X,Z) \} \right.
\]
\[
+ \eta(Y)g(X,Z) \} - \left\{ \eta(Y)\bar{S}(X,Z) + \eta(Z)\bar{S}(X,Y) \right\} \right]
\]
\[
- \alpha \left[ (n-1)\alpha^2 \{\eta(Z)g(\phi X,Y) + \eta(Y)g(\phi X,Z) \} \right.
\]
\[
- \{\eta(Y)\bar{S}(\phi X,Z) + \eta(Z)\bar{S}(\phi X,Y) \} \right]  
\]
\[
- \frac{1}{n-2} \left[ (n-1)\alpha^2 \{\eta(Z)\bar{S}(X,Y) + \eta(Y)\bar{S}(X,Z) \} \right.
\]
\[
- \{\bar{S}^2(X,Z)\eta(Y) + \bar{S}^2(X,Y)\eta(Z) \} \right] = 0. 
\]
Finally, replacing \( Z = \xi \) in (33) and using (23) and (24), we get (29). \( \square \)

An \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold \( M \) with a semi-symmetric metric connection \( D \) is said to be \( \eta \)-Einstein if its Ricci tensor \( \bar{S} \) is of the form
\[
\bar{S}(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y),
\]
where \( A, B \) are smooth functions on \( M \). We consider the vector fields \( e_i, i = 1, 2, \ldots, n \), which forms an orthonormal basis for the tangent space \( T_x M \) of \( M \).

Now, putting \( X = Y = e_i, i = 1, 2, \ldots, n, \) in (35) and summing over \( i = 1, \ldots, n, \) we get
\[
An - B = \bar{r}. 
\]
Again, replacing \( X = Y = \xi \) in (35), we have that
\[
A - B = (n-1)\alpha^2. 
\]
Solving (36) and (37), we obtain
\[
A = \frac{\bar{r}}{n-1} - \alpha^2 \quad \text{and} \quad B = \frac{\bar{r}}{n-1} - n\alpha^2. 
\]
Thus the Ricci tensor of an \( \eta \)-Einstein manifold with a semi-symmetric metric connection \( D \) is given by
\[
\bar{S}(X,Y) = \left[ \frac{\bar{r}}{n-1} - \alpha^2 \right] g(X,Y) + \left[ \frac{\bar{r}}{n-1} - n\alpha^2 \right] \eta(X)\eta(Y). 
\]

6. \( \eta \)-Einstein Lorentzian \( \alpha \)-Sasakian manifold with a semi-symmetric metric connection \( D \) satisfying the condition \( \bar{C}.\bar{S} = 0 \)

**Theorem 6.1.** Let \( M \) be an \( \eta \)-Einstein Lorentzian \( \alpha \)-Sasakian manifold of dimension \( n \) with the restriction \( X = Z = \xi \). Then \( \bar{C}.\bar{S} = 0 \) if and only if
\[
g(\phi Y,\phi W) = -\alpha g(\phi Y,W), \quad Y,W \in \chi(M). 
\]
Proof. Let $M$ be an $\eta$-Einstein Lorentzian $\alpha$-Sasakian manifold of the semi-symmetric metric connection $D$ satisfying $\bar{C}.\bar{S} = 0$. Using (38) in (30), we get
\[ \eta(\bar{C}(X,Y)Z)\eta(W) + \eta(\bar{C}(X,Y)W)\eta(Z) = 0. \]
Further, using (16), (21) and (23) in the above equation, we obtain that
\[
\{ g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\
+ g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z) \\
- \alpha\{ g(\phi Y,Z)\eta(X)\eta(W) - g(\phi X,Z)\eta(Y)\eta(W) \\
+ g(\phi Y,W)\eta(X)\eta(Z) - g(\phi X,W)\eta(Y)\eta(Z) \} = 0.
\]
Putting here $X = Z = \xi$, we get
\[ g(\phi Y,\phi W) = -\alpha g(\phi Y,W). \]
Conversely,
\[ \bar{C}.\bar{S} = \{ g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\
+ g(Y,W)\eta(X)\eta(Z) - g(X,W)\eta(Y)\eta(Z) \\
- \alpha\{ g(\phi Y,Z)\eta(X)\eta(W) - g(\phi X,Z)\eta(Y)\eta(W) \\
+ g(\phi Y,W)\eta(X)\eta(Z) - g(\phi X,W)\eta(Y)\eta(Z) \}. \]
Using $X = Z = \xi$ in this equation, we get
\[ \bar{C}.\bar{S} = g(Y,W) + \eta(Y)\eta(W) + \alpha g(\phi Y,W). \]
Thus $\bar{C}.\bar{S} = 0$. $\square$

7. Ricci pseudosymmetric Lorentzian $\alpha$-Sasakian manifold with a semi-symmetric metric connection $D$

**Theorem 7.1.** A Ricci pseudosymmetric Lorentzian $\alpha$-Sasakian manifold $M$ with a semi-symmetric metric connection $D$ and with restrictions $Y = W = \xi$, $L_{\bar{S}} = \alpha^2$ is an Einstein manifold.

**Proof.** Recall that a Lorentzian $\alpha$-Sasakian manifold $M$ with a semi-symmetric metric connection $D$ is called Ricci pseudosymmetric if
\[ (\bar{R}(X,Y).\bar{S})(Z,W) = L_{\bar{S}} \left[ ((X \wedge Y).\bar{S}))(Z,W) \right] \]
or
\[ \bar{S}(\bar{R}(X,Y)Z,W) + \bar{S}(Z,\bar{R}(X,Y)W) = L_{\bar{S}} \left[ \bar{S}((X \wedge Y)Z,W) + \bar{S}(Z,(X \wedge Y)W) \right]. \]  
Putting $Y = W = \xi$, in (39) and using (1), (20) and (23), we have
\[
\begin{align*}
[L_{\bar{S}} - \alpha^2] \left[ \bar{S}(Z,X) - (n-1)\alpha^2 g(Z,X) \right] \\
= -\alpha \left[ \bar{S}(Z,\phi X) - (n-1)\alpha^2 g(Z,\phi X) \right].
\end{align*}
\]
Then, for $L_{\bar{S}} = \alpha^2$,
\[ \bar{S}(Z, \phi X) = (n - 1)\alpha^2 g(Z, \phi X). \]
Thus $M$ is an Einstein manifold.

**Corollary 7.1.** If $M$ is a Ricci pseudosymmetric Lorentzian $\alpha$-Sasakian manifold with a semi-symmetric metric connection $D$ and with restriction $Y = W = \xi$, then
\[ \alpha \left[ \bar{S}(Z, X) - (n - 1)\alpha^2 g(Z, X) \right] = \bar{S}(Z, \phi X) - (n - 1)\alpha^2 g(Z, \phi X). \]

**Proof.** If $M$ is a Ricci pseudosymmetric Lorentzian $\alpha$-Sasakian manifold with a semi-symmetric metric connection $D$, then $L_{\bar{S}} = 0$. Putting $L_{\bar{S}} = 0$ in (40), we get (41).

\[ \blacksquare \]

8. Pseudosymmetric Lorentzian $\alpha$-Sasakian manifold and Weyl pseudosymmetric Lorentzian $\alpha$-Sasakian manifold with semi-symmetric metric connections

In the present section, we shall give the definitions of a pseudosymmetric and a Weyl-pseudosymmetric Lorentzian $\alpha$-Sasakian manifolds with semi-symmetric metric connections and discuss their properties.

**Definition 8.1.** A Lorentzian $\alpha$-Sasakian manifold $M$ with a semi-symmetric metric connection $D$ is said to be *pseudosymmetric* if the curvature tensor $\bar{R}$ of $D$ satisfies the condition
\[ ((\bar{R}(X, Y)).\bar{R})(U, V, W) = L_{\bar{R}} \left[ ((X \wedge Y)).\bar{R})(U, V, W) \right], \tag{42} \]
where
\[ ((\bar{R}(X, Y)).\bar{R})(U, V, W) = \bar{R}(X, Y)(\bar{R}(U, V)W) - \bar{R}(\bar{R}(X, Y)U, V)W - \bar{R}(U, \bar{R}(X, Y)V)W - \bar{R}(U, V)\bar{R}(X, Y)W \tag{43} \]
and
\[ ((X \wedge Y)).\bar{R})(U, V, W) = (X \wedge Y)(\bar{R}(U, V)W) - \bar{R}((X \wedge Y)U, V)W - \bar{R}(U, (X \wedge Y)V)W - \bar{R}(U, V)((X \wedge Y)W). \tag{44} \]

**Definition 8.2.** A Lorentzian $\alpha$-Sasakian manifold $M$ with a semi-symmetric metric connection $D$ is said to be *Weyl pseudosymmetric* if the curvature tensor $\bar{R}$ of $D$ satisfies the condition
\[ ((\bar{R}(X, Y)).\bar{C})(U, V, W) = L_{\bar{C}} \left[ ((X \wedge Y)).\bar{C})(U, V, W) \right], \tag{45} \]
where
\[ ((\bar{R}(X, Y)).\bar{C})(U, V, W) = \bar{R}(X, Y)(\bar{C}(U, V)W) - \bar{C}(\bar{R}(X, Y)U, V)W - \bar{C}(U, \bar{R}(X, Y)V)W - \bar{C}(U, V)\bar{R}(X, Y)W \tag{46} \]
and

\[(X \wedge Y)\tilde{C}(U, V, W) = (X \wedge Y)(\tilde{C}(U, V)W) - \tilde{C}((X \wedge Y)U, V)W \]
\[ - \tilde{C}(U, (X \wedge Y)V)W - \tilde{C}(U, V)((X \wedge Y)W). \] (47)

**Theorem 8.1.** Let \( M \) be an \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold. If \( M \) is Weyl pseudosymmetric, then \( M \) is either conformally flat or \( L_C = \alpha^2 \).

**Proof.** Let \( M \) be Weyl pseudosymmetric and \( X, Y, U, V, W \in \chi(M) \). Then, using (45) and (46) in (44), we have

\[
\tilde{R}(X, Y)(\tilde{C}(U, V)W) - \tilde{C}(\tilde{R}(X, Y)U, V)W \\
- \tilde{C}(U, \tilde{R}(X, Y)V)W - \tilde{C}(U, V)(\tilde{R}(X, Y)W) \\
= L_C \left[ (X \wedge Y)(\tilde{C}(U, V)W) - \tilde{C}((X \wedge Y)U, V)W \\
- \tilde{C}(U, (X \wedge Y)V)W - \tilde{C}(U, V)((X \wedge Y)W) \right].
\] (48)

Replacing here \( X \) with \( \xi \), we obtain

\[
\tilde{R}(\xi, Y)(\tilde{C}(U, V)W) - \tilde{C}(\tilde{R}(\xi, Y)U, V)W \\
- \tilde{C}(U, \tilde{R}(\xi, Y)V)W - \tilde{C}(U, V)(\tilde{R}(\xi, Y)W) \\
= L_C \left[ (\xi \wedge Y)(\tilde{C}(U, V)W) - \tilde{C}((\xi \wedge Y)U, V)W \\
- \tilde{C}(U, (\xi \wedge Y)V)W - \tilde{C}(U, V)((\xi \wedge Y)W) \right].
\]

Using (1) and (20) in (47), and taking inner product of (47) with \( \xi \), we get

\[\alpha^2 \left[ -\tilde{C}(U, V, W, Y) - \eta(\tilde{C}(U, V)W)\eta(Y) \right.\]
\[+ g(Y, U)\eta(\tilde{C}(\xi, V)W) - \eta(U)\eta(\tilde{C}(Y, V)W) \]
\[+ g(Y, V)\eta(\tilde{C}(U, \xi)W) - \eta(V)\eta(\tilde{C}(U, Y)W) - \eta(W)\eta(\tilde{C}(U, V)Y) \]
\[+ \alpha \left[ \tilde{C}(U, V, W, \phi Y) + \eta(U)\eta(\tilde{C}(\phi Y, V)W) - g(\phi Y, U)\eta(\tilde{C}(\xi, V)W) \right.\]
\[+ \eta(Y)\eta(\tilde{C}(U, \phi Y)W) - g(\phi Y, V)\eta(\tilde{C}(U, \xi)W) + \eta(W)\eta(\tilde{C}(U, \phi Y)W) \]
\[= L_C \left[ -\tilde{C}(U, V, W, Y) - \eta(Y)\eta(\tilde{C}(U, V)W) + g(Y, U)\eta(\tilde{C}(\xi, V)W) \right.\]
\[+ \eta(U)\eta(\tilde{C}(Y, V)W) + g(Y, V)\eta(\tilde{C}(U, \xi)W) \]
\[+ \eta(Y)\eta(\tilde{C}(U, \xi)W) - \eta(W)\eta(\tilde{C}(U, Y)W) \right].
\]

Then, putting \( Y = U = \xi \), we get

\[ [L_C - \alpha^2] \eta(\tilde{C}(\xi, V)W) = 0. \]

This shows that either \( \eta(\tilde{C}(\xi, V)W) = 0 \) or \( L_C - \alpha^2 = 0 \).

Now, if \( L_C - \alpha^2 \neq 0 \), then \( \eta(\tilde{C}(\xi, V)W) = 0 \), i.e., \( M \) is conformally flat and

\[ S(V, W) = Ag(V, W) + Bg(V, \eta(W)) - \alpha g(\phi V, W), \]
with
\[ A = \left[ \alpha^2 - \frac{(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)} \right] (n-2), \]
and
\[ B = \left[ \alpha^2 - 2\frac{(n-1)\alpha^2}{n-2} + \frac{\bar{r}}{(n-1)(n-2)} \right] (n-2). \]

But if \( \eta(\bar{C}(\xi, V)W) \neq 0 \), then we have \( L_{\bar{C}} = \alpha^2 \).

**Theorem 8.2.** Let \( M \) be an \( n \)-dimensional Lorentzian \( \alpha \)-Sasakian manifold. If \( M \) is pseudosymmetric, then either \( M \) is a space of constant curvature and \( F(X, Y) = \alpha g(\phi X, \phi Y) \) for \( \alpha \neq 0 \), or \( L_{\bar{R}} = \alpha^2 \) for \( X, Y \in \chi(M) \).

**Proof.** Let \( M \) be pseudosymmetric and let \( X, Y, U, V, W \in \chi(M) \). Then, using (42) and (43) in (41), we have that
\[
\bar{R}(X, Y)(\bar{R}(U, V)W) - \bar{R}(\bar{R}(X, Y)U, V)W
- \bar{R}(U, \bar{R}(X, Y)V)W - \bar{R}(U, V)(\bar{R}(X, Y)W)
= L_{\bar{R}} \left[(X \wedge Y)(\bar{R}(U, V)W) - \bar{R}((X \wedge Y)U, V)W
- \bar{R}(U, (X \wedge Y)V)W - \bar{R}(U, V)((X \wedge Y)W) \right].
\]
Replacing here \( X \) with \( \xi \), we obtain
\[
\bar{R}(\xi, Y)(\bar{R}(U, V)W) - \bar{R}(\bar{R}(\xi, Y)U, V)W
- \bar{R}(U, \bar{R}(\xi, Y)V)W - \bar{R}(U, V)(\bar{R}(\xi, Y)W)
= L_{\bar{R}} \left[(\xi \wedge Y)(\bar{R}(U, V)W) - \bar{R}((\xi \wedge Y)U, V)W
- \bar{R}(U, (\xi \wedge Y)V)W - \bar{R}(U, V)((\xi \wedge Y)W) \right].
\]
Using (1), (20) in (48) and taking inner product of (48) with \( \xi \), we get
\[
\alpha^2 \left[-\bar{R}(U, V, W, Y) - \eta(\bar{R}(U, V)W)\eta(Y)
+ g(Y, U)\eta(\bar{R}(\xi, V)W) - \eta(U)\eta(\bar{R}(Y, V)W)
+ g(Y, V)\eta(\bar{R}(U, \xi)W) - \eta(V)\eta(\bar{R}(U, Y)W) - \eta(W)\eta(\bar{R}(U, V)Y) \right]
+ \alpha \left[ \bar{R}(U, V, W, \phi Y) + \eta(U)\eta(\bar{R}(\phi Y, V)W) - g(\phi Y, U)\eta(\bar{R}(\xi, V)W)
+ \eta(V)\eta(\bar{R}(U, \phi Y)W) - g(\phi Y, V)\eta(\bar{R}(U, \xi)W) + \eta(W)\eta(\bar{R}(U, \phi Y)W) \right]
= L_{\bar{R}} \left[ -\bar{R}(U, V, W, Y) - \eta(Y)\eta(\bar{R}(U, V)W) + g(Y, U)\eta(\bar{R}(\xi, V)W)
- \eta(U)\eta(\bar{R}(Y, V)W) + g(Y, V)\eta(\bar{R}(U, \xi)W)
- \eta(V)\eta(\bar{R}(U, Y)W) - \eta(W)\eta(\bar{R}(U, V)Y) \right].
\]
Then, putting \( Y = U = \xi \), we get
\[
\left[ L_{\bar{R}} - \alpha^2 \right] \eta(\bar{R}(\xi, V)W) = 0.
\]
This shows that either \( \eta(\bar{R}(\xi, V)W) = 0 \) or \( L_{\bar{R}} - \alpha^2 = 0 \).
Now, if $L_\bar{R} - \alpha^2 \neq 0$, then $\eta(\bar{R}(\xi, V)W) = 0$ which implies that $M$ is a space of constant curvature and

\[
\alpha g(\phi V, \phi W) = g(\phi V, W)
\]
or

\[
F(V, W) = \alpha g(\phi V, \phi W).
\]

If $\eta(\bar{R}(\xi, V)W) \neq 0$, then we have $L_\bar{R} = \alpha^2$. □

9. Example of a pseudosymmetric Lorentzian $\alpha$-Sasakian manifold with a semi-symmetric metric connection $D$

Let us consider a three-dimensional manifold

\[
M = \{(x_1, x_2, x_3) \in R^3 : x_1, x_2, x_3 \in R\},
\]

where $(x_1, x_2, x_3)$ are the standard coordinates of $R^3$. We consider the vector fields

\[
e_1 = e^{x_3} \frac{\partial}{\partial x_2}, \quad e_2 = e^{x_3}(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}) \quad \text{and} \quad e_3 = \alpha \frac{\partial}{\partial x_3},
\]

where $\alpha$ is a constant.

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vector fields for each point of $M$ and hence a basis of $T_x M$. The Lorentzian metric $g$ is defined by

\[
g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = -1.
\]

Then the form of metric becomes

\[
g = -\frac{1}{(e^{x_3})^2}(dx_2)^2 - \frac{1}{\alpha^2}(dx_3)^2
\]

which is a Lorentzian metric.

Let $\eta$ be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$, and let $\phi$ be the $(1, 1)$-tensor field defined by

\[
\phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0.
\]

From the linearity of $\phi$ and $g$, we have that

\[
\eta(e_3) = -1, \quad \phi^2(X) = X + \eta(X) e_3 \quad \text{and} \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)
\]

for any $X \in \chi(M)$. Then, for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines a Lorentzian paracontact structure on $M$.

Let $\nabla$ be the Levi–Civita connection of the Lorentzian metric $g$. Then

\[
[e_1, e_2] = 0, \quad [e_1, e_3] = -\alpha e_1, \quad [e_2, e_3] = -\alpha e_2.
\]

Recall Koszul’s formula:

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).
\]
From the above formula, we can calculate the following:
\[
\nabla_{e_1} e_1 = -\alpha e_3, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -\alpha e_1, \\
\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -\alpha e_3, \quad \nabla_{e_2} e_3 = -\alpha e_2, \\
\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.
\]

Hence the structure \((\phi, \xi, \eta, g)\) is a Lorentzian \(\alpha\)-Sasakian manifold (see [11]).

Using (9), we find \(D\), the semi-symmetric metric connection on \(M\):
\[
D_{e_1} e_1 = (1 - \alpha)e_3, \quad +, D_{e_1} e_2 = 0, \quad D_{e_1} e_3 = -(1 + \alpha)e_1, \\
D_{e_2} e_1 = 0, \quad D_{e_2} e_2 = (1 - \alpha)e_3, \quad D_{e_2} e_3 = -(1 + \alpha)e_1, \\
D_{e_3} e_1 = 0, \quad D_{e_3} e_2 = 0, \quad D_{e_3} e_3 = 0.
\]

Using (10), the torson tensor \(\tilde{T}\) of the semi-symmetric metric connection \(D\) may be expressed as follows:
\[
\tilde{T}(e_i, e_i) = 0, \quad i = 1, 2, 3, \\
\tilde{T}(e_1, e_2) = 0, \quad \tilde{T}(e_1, e_3) = -e_1, \quad \tilde{T}(e_2, e_3) = -e_2.
\]

Also,
\[
(D_{e_1} g)(e_2, e_3) = (D_{e_2} g)(e_3, e_1) = (D_{e_3} g)(e_1, e_2) = 0.
\]

Thus \(M\) is a Lorentzian \(\alpha\)-Sasakian manifold with a semi-symmetric metric connection \(D\).

Now, we calculate the curvature tensor \(\bar{R}\) and the Ricci tensor \(\bar{S}\) as follows:
\[
\bar{R}(e_1, e_2)e_3 = 0, \quad \bar{R}(e_1, e_3)e_3 = -(\alpha^2 + \alpha)e_1, \\
\bar{R}(e_3, e_2)e_2 = (\alpha^2 - \alpha)e_3, \quad \bar{R}(e_3, e_1)e_1 = (\alpha^2 - \alpha)e_3, \\
\bar{R}(e_2, e_1)e_1 = (\alpha^2 - 2\alpha - 1)e_2, \quad \bar{R}(e_2, e_3)e_3 = -(\alpha^2 + \alpha)e_2, \\
\bar{R}(e_1, e_2)e_2 = (\alpha^2 - 2\alpha - 1)e_1, \quad \bar{S}(e_3, e_3) = -2\alpha^2, \\
\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = -(n - 2)(\alpha + 1).
\]

Again, using (1), we get
\[
(e_1, e_2)e_3 = 0, \quad (e_i \wedge e_i) e_j = 0, \quad i, j = 1, 2, 3, \\
(e_1 \wedge e_2)e_2 = (e_1 \wedge e_3)e_3 = -e_1, \quad (e_2 \wedge e_1)e_1 = (e_2 \wedge e_3)e_3 = -e_2, \\
(e_3 \wedge e_2)e_2 = (e_3 \wedge e_1)e_1 = -e_3.
\]

Now,
\[
\bar{R}(e_1, e_2)(\bar{R}(e_3, e_1)e_2) = 0, \quad \bar{R}(\bar{R}(e_1, e_2)e_3, e_1)e_2 = 0, \\
\bar{R}(e_3, \bar{R}(e_1, e_2)e_1)e_2 = (1 + 2\alpha - \alpha^2)(\alpha^2 - \alpha)e_3, \\
(\bar{R}(e_3, e_1)(\bar{R}(e_1, e_2)e_2) = (\alpha^2 - 2\alpha - 1)(\alpha^2 - \alpha)e_3.
\]

Then \((\bar{R}(e_1, e_2), \bar{R})(e_3, e_1, e_2) = 0.\)
Again,

\[(e_1 \wedge e_2)(\bar{R}(e_3, e_1)e_2) = 0,\]
\[\bar{R}((e_1 \wedge e_2)e_3, e_1)e_2 = 0,\]
\[\bar{R}(e_3, (e_1 \wedge e_2)e_1)e_2 = (\alpha^2 - \alpha)e_3,\]
\[\bar{R}(e_3, e_1)((e_1 \wedge e_2)e_2) = (\alpha - \alpha^2)e_3.\]

Consequently, \(((e_1, e_2)\bar{R})(e_3, e_1, e_2) = 0.\) Thus

\[(\bar{R}(e_1, e_2), \bar{R})(e_3, e_1, e_2) = L_{\bar{R}} \left[ ((e_1, e_2)\bar{R})(e_3, e_1, e_2) \right]\]

for any function \(L_{\bar{R}} \in C^\infty(M)\). Similarly, for any combination of \(e_1, e_2\) and \(e_3\), we can show (45). Hence \(M\) is a pseudosymmetric Lorentzian \(\alpha\)-Sasakian manifold with semi-symmetric metric connection.

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