A note on spark varieties

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Abstract. We study basic properties (e.g., algebraicity, reducibility, and dimension) of certain sets of matrices defined by means of the spark.

0. Introduction and preliminaries

Throughout the text \( \mathbb{F} \) stands for a field, and \( \mathbb{N} \) for the set of positive integers. The present paper is about certain algebraic subsets of \( \mathcal{M}_{m \times n}(\mathbb{F}) \), the vector space of all \( m \times n \) matrices over \( \mathbb{F} \). We start by recalling a few definitions and properties.

Consider a nonzero finite dimensional vector space \( V \) over \( \mathbb{F} \) and a linear isomorphism \( \varphi : V \rightarrow \mathbb{F}^d \), where \( d = \dim_{\mathbb{F}} V \). A set \( E \subseteq V \) is said to be algebraic, if

\[
\exists s \in \mathbb{N} \exists f_1, \ldots, f_s \in \mathbb{F}[x_1, \ldots, x_d] : E = \{ v \in V : f_1(\varphi(v)) = \ldots = f_s(\varphi(v)) = 0 \}.
\]

A set \( Q \subseteq V \) is said to be quasi-algebraic, if \( Q = E_1 \setminus E_2 \) for some algebraic sets \( E_1, E_2 \subseteq V \) (i.e., if it is locally closed in the Zariski topology on \( V \)).

The linear capacity of an algebraic set \( E \subseteq V \) is defined by

\[
\Lambda(E) = \sup \{ \dim_{\mathbb{F}} L : L \text{ is a linear subspace of } V, L \subseteq E \}.
\]

The linear capacity was introduced in [5]. The following properties are quite obvious (cf. [5, Proposition 1.1]).

**Proposition 0.1.** Let \( V \) and \( W \) be nonzero finite dimensional vector spaces over \( \mathbb{F} \), let \( E_1 \subseteq V \) and \( E_2, G \subseteq W \) be algebraic sets, and let \( \psi : V \rightarrow W \) be a linear map such that \( \psi(E_1) \subseteq G \). Then

(i) \( \Lambda(E_1) = -\infty \) if and only if the zero vector does not belong to \( E_1 \),

(ii) \( \Lambda(E_1 \times E_2) = \Lambda(E_1) + \Lambda(E_2) \).
(iii) \( \Lambda(E_1) \leq \Lambda(G) \) whenever \( E_1 \neq \emptyset \) and the restriction \( \psi|_{E_1} \) is injective.

(iv) \( \Lambda(E_1) = \Lambda(G) \) whenever \( E_1 \neq \emptyset, \psi|_{E_1} : E_1 \to G \) is bijective, and \((\psi|_{E_1})^{-1}\) is the restriction of a linear map \( \xi : W \to V \).

Finally, let us assume that the field \( F \) is algebraically closed. An algebraic set \( E \subseteq V \) is said to be normal, if it is irreducible and the coordinate ring \( F[E] \) is integrally closed in the function field \( F(E) \). We refer to [3] for more information on algebraic geometry.

Let \( m, n \in \mathbb{N} \) and \( r \) be a non-negative integer such that \( r \leq \min\{m, n\} \).

The generic determinantal variety \( \mathcal{H}^r_{m \times n} \) defined by

\[
\mathcal{H}^r_{m \times n} = \{ A \in \mathcal{M}_{m \times n}(F) : \text{rank}(A) \leq r \}
\]

is an important classical example of an algebraic subset of \( \mathcal{M}_{m \times n}(F) \). The Flanders–Meshulam theorem [4] says that \( \Lambda(\mathcal{H}^r_{m \times n}) = r \max\{m, n\} \). Moreover, it is well known [1] that if the field \( F \) is algebraically closed, then \( \mathcal{H}^r_{m \times n} \) is normal and \( \dim \mathcal{H}^r_{m \times n} = r(m + n - r) \).

In [2], Donoho and Elad introduced the notion of spark of a matrix.

**Definition 0.2.** Let \( C_1, \ldots, C_n \in F^m \) be the columns of a matrix \( A \in \mathcal{M}_{m \times n}(F) \). The spark of \( A \) is defined to be the infimum of the set of all positive integers \( \ell \) such that

\[
\exists j_1, \ldots, j_\ell \in \{1, \ldots, n\} : \left\{ j_1 < \ldots < j_\ell, C_{j_1}, \ldots, C_{j_\ell} \text{ are linearly dependent} \right\}.
\]

The definition of the spark is similar to the definition of the rank of a matrix. However, algebraic and computational properties of the spark are very different from those of the rank.

In the present note we will look at the notion of spark of a matrix from a geometric point of view. Namely, for a positive integer \( k \) we define the spark variety \( \mathcal{S}^k_{m \times n} \) by

\[
\mathcal{S}^k_{m \times n} = \{ A \in \mathcal{M}_{m \times n}(F) : \text{spark}(A) \leq k \},
\]

and our goal is to give a geometric characterization of the sets \( \mathcal{S}^k_{m \times n} \) (analogous to the characterization of the generic determinantal varieties).

### 1. Basic properties

First, let us collect some remarks about the family of all spark varieties in \( \mathcal{M}_{m \times n}(F) \).

**Proposition 1.1.** (i) The zero matrix belongs to \( \mathcal{S}^k_{m \times n} \) and \( \lambda \mathcal{S}^k_{m \times n} \subseteq \mathcal{S}^k_{m \times n} \) for all \( \lambda \in F \).

(ii) \( \mathcal{S}^k_{m \times n} \subseteq \mathcal{S}^{k+1}_{m \times n} \).

(iii) If \( m < n \), then \( \mathcal{S}^{m+1}_{m \times n} = \mathcal{M}_{m \times n}(F) \).
(iv) If \( r \in \mathbb{N} \cup \{0\} \) is such that \( r \leq m \) and \( r < n \), then \( \mathcal{H}_{m \times n}^r \subseteq \mathcal{S}_{m \times n}^{r+1} \).

(v) If \( \min\{k, m\} \geq n \), then \( S_{m \times n}^k = \mathcal{H}_{m \times n}^{n-1} \).

**Proof.** Observe that the spark of the zero matrix is equal to 1. Moreover,
\[
\forall A \in \mathcal{M}_{m \times n}(\mathbb{F}) \forall \lambda \in \mathbb{F} \setminus \{0\} : \text{spark}(\lambda A) = \text{spark}(A).
\]

Property (i) follows. Inclusion (ii) is obvious.

It is easy to see that
\[
\forall A \in \mathcal{M}_{m \times n}(\mathbb{F}) : \begin{cases} \text{spark}(A) \neq +\infty \Rightarrow \text{spark}(A) \leq \text{rank}(A) + 1, \\ \text{spark}(A) = +\infty \iff \text{rank}(A) = n. \end{cases}
\]

Consequently, if \( m < n \), then \( \text{spark}(A) \leq m + 1 \) for all \( A \in \mathcal{M}_{m \times n}(\mathbb{F}) \).

Property (iii) follows.

If \( r \in \mathbb{N} \cup \{0\} \), \( r \leq m \), \( r < n \), and \( A \in \mathcal{H}_{m \times n}^r \), then \( \text{spark}(A) \leq r + 1 \). This yields property (iv).

Suppose finally that \( \min\{k, m\} \geq n \). Then, by (iv) and (ii), we have
\[
\mathcal{H}_{m \times n}^{n-1} \subseteq \mathcal{S}_{m \times n}^n \subseteq \mathcal{S}_{m \times n}^k.
\]

On the other hand,
\[
\forall A \in \mathcal{M}_{m \times n}(\mathbb{F}) : \text{spark}(A) \neq +\infty \iff \text{rank}(A) \leq n - 1,
\]

and hence \( S_{m \times n}^k \subseteq \mathcal{H}_{m \times n}^{n-1} \). Property (v) follows.

Let \( m, n, \ell \in \mathbb{N} \) be such that \( \ell \leq \min\{m, n\} \). For \( A = [a_{ij}] \in \mathcal{M}_{m \times n}(\mathbb{F}) \), a strictly increasing sequence \((i_1, \ldots, i_\ell)\) of elements of \( \{1, \ldots, m\} \) and a strictly increasing sequence \((j_1, \ldots, j_\ell)\) of elements of \( \{1, \ldots, n\} \) we define \( \mu_{j_1, \ldots, j_\ell}(A) \) to be the determinant of the matrix \([a_{i_j k}] \in \mathcal{M}_{\ell \times \ell}(\mathbb{F}) \).

**Theorem 1.2.** Every spark variety \( S_{m \times n}^k \) is an algebraic subset of the space \( \mathcal{M}_{m \times n}(\mathbb{F}) \).

**Proof.** If \( \ell := \min\{k, n\} > m \), then by Proposition 1.1 we have \( S_{m \times n}^k = \mathcal{M}_{m \times n}(\mathbb{F}) \). Suppose therefore that \( \ell \leq m \). Let \((j_1, \ldots, j_\ell)\) be a strictly increasing sequence of elements of \( \{1, \ldots, n\} \). We define
\[
\mathcal{D}_{j_1, \ldots, j_\ell} = \left\{ A \in \mathcal{M}_{m \times n}(\mathbb{F}) : \mu_{j_1, \ldots, j_\ell}^{i_1, \ldots, i_\ell}(A) = 0 \text{ for all } i_1, \ldots, i_\ell \in \{1, \ldots, m\} \text{ such that } i_1 < \ldots < i_\ell \right\}.
\]

Notice that \( \mathcal{D}_{j_1, \ldots, j_\ell} \) is an algebraic subset of \( \mathcal{M}_{m \times n}(\mathbb{F}) \). Moreover, \( \mathcal{D}_{j_1, \ldots, j_\ell} \) is equal to the totality of matrices in \( \mathcal{M}_{m \times n}(\mathbb{F}) \) whose columns with indices \( j_1, \ldots, j_\ell \) are linearly dependent. If \( s \in \{1, \ldots, \ell\} \), then each \( s \)-element set of linearly dependent columns of a matrix \( A \in \mathcal{M}_{m \times n}(\mathbb{F}) \) is contained in an \( \ell \)-element set of linearly dependent columns of \( A \). Thus,
\[
S_{m \times n}^k = S_{m \times n}^\ell = \bigcup \{ \mathcal{D}_{j_1, \ldots, j_\ell} : j_1, \ldots, j_\ell \in \{1, \ldots, n\}, j_1 < \ldots < j_\ell \}.
\]

The algebraicity follows. 

\( \square \)
The sets $D_{j_1,\ldots,j_k}$ are examples of so-called linear determinantal varieties.

**Corollary 1.3.** (i) For any $k \in \mathbb{N} \cup \{+\infty\}$, the set $\{A \in M_{m \times n}(\mathbb{F}) : \text{spark}(A) = k\}$ is quasi-algebraic.
(ii) If $m < n$, then $\{A \in M_{m \times n}(\mathbb{F}) : \text{spark}(A) = m + 1\}$ is open in the Zariski topology on $M_{m \times n}(\mathbb{F})$.

Recall that if $m < n$, then $\text{spark}(A) \leq m + 1$ for all $A \in M_{m \times n}(\mathbb{F})$. Moreover, if $m \geq n$, then $\text{max}_{A \in M_{m \times n}(\mathbb{F})} \text{spark}(A) = +\infty$ and the set
\[
\{A \in M_{m \times n}(\mathbb{F}) : \text{spark}(A) = +\infty\} = M_{m \times n}(\mathbb{F}) \setminus H_{m \times n}^{-1}
\]
is open in the Zariski topology on $M_{m \times n}(\mathbb{F})$.

2. Main results

We are in a position to describe the geometric structure of a spark variety.

**Theorem 2.1.** Suppose that the field $\mathbb{F}$ is algebraically closed. Let $m, n, k \in \mathbb{N}$ be such that $k \leq m$ and $k < n$. Then the family of all irreducible components of $S_{m \times n}^k$ coincides with
\[
\{D_{j_1,\ldots,j_k} : j_1, \ldots, j_k \in \{1, \ldots, n\}, j_1 < \ldots < j_k\}.
\]
Moreover,
- the above sets $D_{j_1,\ldots,j_k}$ are normal and have dimension $m(n - 1) + k - 1$,
- $\bigcap \{D_{j_1,\ldots,j_k} : j_1, \ldots, j_k \in \{1, \ldots, n\}, j_1 < \ldots < j_k\} = H_{m \times n}^{k-1}$.

**Proof.** Recall from the proof of Theorem 1.2 that
\[
S_{m \times n}^k = \bigcup \{D_{j_1,\ldots,j_k} : j_1, \ldots, j_k \in \{1, \ldots, n\}, j_1 < \ldots < j_k\}
\]
and the sets $D_{j_1,\ldots,j_k}$ are algebraic.

Let $(j_1',\ldots,j_k')$ and $(j_1'',\ldots,j_k'')$ be two distinct strictly increasing sequences of elements of $\{1, \ldots, n\}$. Since $k \leq m$, there exists a matrix in $M_{m \times n}(\mathbb{F})$ such that its columns with indices $j_1', \ldots, j_k'$ are linearly independent while its columns with indices belonging to $\{j_1', \ldots, j_k'\} \setminus \{j_1'', \ldots, j_k''\}$ are not. Thus, $D_{j_1',\ldots,j_k'}$ is not contained in $D_{j_1'',\ldots,j_k''}$.

Pick a strictly increasing sequence $(j_1, \ldots, j_k)$ of elements of $\{1, \ldots, n\}$. Define $A' \in M_{m \times k}(\mathbb{F})$ to be the matrix that consists of the columns of a matrix $A \in M_{m \times n}(\mathbb{F})$ with indices $j_1, \ldots, j_k$, and $A'' \in M_{m \times (n-k)}(\mathbb{F})$ to be the matrix that consists of all other columns of $A$. The map
\[
D_{j_1,\ldots,j_k} \ni A \longrightarrow (A', A'') \in H_{m \times k}^{k-1} \times M_{m \times (n-k)}(\mathbb{F})
\]
is an isomorphism of algebraic sets. Therefore, since $H_{m \times k}^{k-1}$ is normal, so is $D_{j_1,\ldots,j_k}$. (In particular, $D_{j_1,\ldots,j_k}$ is irreducible.) Moreover, since $\dim H_{m \times k}^{k-1} = \ldots$
(k - 1)(m + k - k + 1), we have
\[
\dim D_{j_1, \ldots, j_k} = \dim H_{m \times k}^{k-1} + \dim M_{m \times (n-k)}(F)
\]
\[
= (k - 1)(m + 1) + m(n - k)
\]
\[
= m(n - 1) + k - 1.
\]

Finally, a matrix \( A \) belongs to all components \( D_{j_1, \ldots, j_k} \) if and only if every \( k \)-element set of its columns is linearly dependent, which means exactly that \( \text{rank}(A) \leq k - 1 \).

**Corollary 2.2.** Let \( F \) be algebraically closed. Then every spark variety \( S^k_{m \times n} \) is pure dimensional and all its irreducible components are normal. Moreover, \( S^k_{m \times n} \) is irreducible if and only if \( k > m \) or \( k \geq n \).

**Proof.** If \( \min\{k, n\} > m \), then by Proposition 1.1 we have that \( S^k_{m \times n} = M_{m \times n}(F) \). Similarly, if \( \min\{k, m\} \geq n \), then \( S^k_{m \times n} = H_{n \times m}^{k-1} \). Thus, \( S^k_{m \times n} \) is normal whenever \( k > m \) or \( k \geq n \). On the other hand, by Theorem 2.1, if \( k \leq m \) and \( k < n \), then \( S^k_{m \times n} \) is reducible and pure dimensional, and all its components are normal.

We will conclude the note by the formula for the linear capacity of spark varieties.

**Lemma 2.3.** Suppose that the field \( F \) is infinite. Let \( V \) be a nonzero finite dimensional vector space over \( F \), let \( s \in \mathbb{N} \), and let \( E_1, \ldots, E_s \subseteq V \) be algebraic sets. Then \( \Lambda(E_1 \cup \ldots \cup E_s) = \max\{\Lambda(E_1), \ldots, \Lambda(E_s)\} \).

**Proof.** We define \( \lambda = \Lambda(E_1 \cup \ldots \cup E_s) \). Then, obviously, \( \max\{\Lambda(E_1), \ldots, \Lambda(E_s)\} \leq \lambda \). Since \( F \) is infinite, every linear subspace of \( V \) is irreducible. Therefore, if \( L \) is a linear subspace of \( V \) such that \( L \subseteq E_1 \cup \ldots \cup E_s \) and \( \dim_F L = \lambda \), then \( L \subseteq E_{i_0} \) for some \( i_0 \in \{1, \ldots, s\} \), and hence \( \lambda \leq \Lambda(E_{i_0}) \leq \max\{\Lambda(E_1), \ldots, \Lambda(E_s)\} \).

**Theorem 2.4.** Suppose that \( F \) is infinite. Let \( m, n, k \in \mathbb{N} \) be such that \( \min\{k, n\} \leq m \). Then
\[
\Lambda(S^k_{m \times n}) = m(n - 1).
\]
(Recall that \( S^k_{m \times n} = M_{m \times n}(F) \) whenever \( \min\{k, n\} > m \).)

**Proof.** If \( \min\{k, m\} \geq n \), then \( S^k_{m \times n} = H_{m \times n}^{n-1} \), and hence the assertion follows from the Flanders–Meshulam theorem. Let us therefore assume that \( k \leq m \) and \( k < n \). The isomorphism considered in the proof of Theorem 2.1 satisfies the assumptions of Proposition 0.1, (iv). Thus, for an arbitrary strictly increasing sequence \( (j_1, \ldots, j_k) \) of elements of \( \{1, \ldots, n\} \), we have \( \Lambda(D_{j_1, \ldots, j_k}) = \Lambda(H_{m \times k}^{k-1} \times M_{m \times (n-k)}(F)) \). By Proposition 0.1, (ii), and the
Flanders–Meshulam theorem,

\[
\Lambda(H_{m \times k}^{k-1} \times M_{m \times (n-k)}(\mathbb{F})) = \Lambda(H_{m \times k}^{k-1}) + \Lambda(M_{m \times (n-k)}(\mathbb{F}))
\]

\[
= (k - 1) \max\{k, m\} + m(n - k)
\]

\[
= m(n - 1).
\]

Since \( S^k_{m \times n} \) coincides with the union of all sets \( D_{j_1, \ldots, j_k} \), the assertion follows now from Lemma 2.3.

\[\square\]

References


