Abstract. Let \( (X, |\cdot|) \) be a seminormed space, \( \Phi = (\phi_k) \) a sequence of moduli, and \( B \) a sequence of infinite scalar matrices \( B' = (b_{kj}) \). Let \( (\lambda, g_\lambda) \) and \( (\Lambda, g_\Lambda) \) be solid F-seminormed (paranormed) spaces of single and double number sequences, respectively. V. Soomer and E. Kolk proved in 1996–1997 that the set of all scalar sequences \( u = (u_k) \) with \( \Phi(u) = (\phi_k(\|u_k\|)) \in \lambda \) is a linear space which may be topologized by the F-seminorm (paranorm) \( g_{\lambda, \Phi}(u) = g_\lambda(\Phi(u)) \) under certain restrictions on \( \Phi \) or \( (\lambda, g_\lambda) \). We generalize this result to the space of all \( X \)-valued sequences \( x = (x_k) \) with \( \phi_k \left( |B'x| \right) \in \Lambda \), where \( B'x = \sum_j b_{kj}x_j \).

Applications are given in the case when \( \Lambda \) is the strong summability domain of a non-negative matrix method. Our corollaries and critical remarks outline results from more than thirty previous papers by many different authors.

1. Introduction

Let \( \mathbb{N} = \{1, 2, \ldots\} \) and let \( \mathbb{K} \) be the field of real numbers \( \mathbb{R} \) or complex numbers \( \mathbb{C} \). In the following we specify the domains of indices for the symbols \( \lim, \sup, \inf \) and \( \sum \) only if they are different from \( \mathbb{N} \). By \( \iota \) we denote the identity mapping \( \iota(z) = z \). In all definitions which contain infinite series we tacitly assume the convergence of these series.

An F-space is usually understood as a complete metrizable topological vector space over \( \mathbb{K} \). The topology of an F-space \( E \) can be given by an F-norm, i.e., by the functional \( g : E \to \mathbb{R} \) with axioms (see [29], p. 13)

\( (N1) \) \( g(0) = 0, \)

\( (N2) \) \( g(x + y) \leq g(x) + g(y) \) \( (x, y \in E), \)

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A functional \( g \) with axioms (N1)–(N4) is called an F-seminorm. A paranorm on \( E \) is defined as a functional \( g : E \to \mathbb{R} \) satisfying axioms (N1), (N2) and
\[
\begin{align*}
\text{(N3)} \quad |\alpha| \leq 1 & \quad (\alpha \in \mathbb{K}), \ x \in E \implies g(\alpha x) \leq g(x), \\
\text{(N4)} \quad \lim_n \alpha_n = 0 & \quad (\alpha_n \in \mathbb{K}), \ x \in E \implies \lim_n g(\alpha_n x) = 0, \\
\text{(N5)} \quad g(x) = 0 \implies x = 0.
\end{align*}
\]

An F-seminorm (paranorm, seminorm) \( g \) is called total if (N5) holds. So, an F-norm \( (\text{norm}) \) is a total F-seminorm (seminorm).

It is known (see [33], Remark 1) that F-seminorms are precisely the paranorms satisfying axiom (N3).

To avoid confusion with the module \(|\cdot|\), following [33], we will often denote the seminorm of an element \( x \in E \) by \(|x|\).

Let \((X,|\cdot|)\) be a seminormed linear space over \( \mathbb{K} \) and let \( X \) be a sequence of seminormed linear spaces \((X_k,|\cdot|_k)\) \((k \in \mathbb{N})\). Then the set \( s^2(X) \) of all double sequences \( x^2 = (x_{ki}) \), \( x_{ki} \in X_k \) \((k, i \in \mathbb{N})\), and the set \( s(X) \) of all sequences \( x = (x_k) \), \( x_k \in X_k \) \((k \in \mathbb{N})\), equipped with coordinatewise addition and scalar multiplication, are linear spaces (over \( \mathbb{K} \)). Any linear subspace of \( s^2(X) \) is called a generalized double sequence space (GDS space) and any linear subspace of \( s(X) \) is called a generalized sequence space (GS space). If \((X_k,|\cdot|_k) = (X,|\cdot|) \((k \in \mathbb{N})\), then we write \( X \) instead of \( X \).

In the case \( X = \mathbb{K} \) we omit the symbol \( X \) in our notation. So, for example, \( s^2 \) and \( s \) denote the linear spaces of all \( \mathbb{K} \)-valued double sequences \( u^2 = (u_{ki}) \) and single sequences \( u = (u_k) \), respectively. As usual, linear subspaces of \( s^2 \) are called double sequence spaces (DS spaces) and linear subspaces of \( s \) are called sequence spaces. Well-known sequence spaces include the sets \( \ell_\infty, c, c_0 \) and \( \ell^p \) \((p > 0)\) of all bounded, convergent, convergent to zero and absolutely \( p \)-summable number sequences, respectively. Examples of DS spaces are
\[
\mathcal{M} = \{ u_k^2 \in s^2 : \bar{u}_k = \sup_i |u_{ki}| < \infty \quad (k \in \mathbb{N})\},
\]
\[
U\lambda = \{ u^2 \in \mathcal{M} : \bar{u} = (\bar{u}_k) \in \lambda \} \quad (\lambda \in \{\ell_\infty, c, c_0, \ell^p\}).
\]

Let \( \mathbb{R}^+ = [0, \infty) \). The idea of a modulus function was shaped by Nakano [37]. Following Ruckle [44] and Maddox [35] we say that a function \( \phi : \mathbb{R}^+ \to \mathbb{R}^+ \) is a modulus function (or, simply, a modulus), if
\[
\begin{align*}
\text{(M1)} \quad \phi(t) = 0 & \iff t = 0, \\
\text{(M2)} \quad \phi(t + u) & \leq \phi(t) + \phi(u) \quad (t, u \in \mathbb{R}^+) \quad \text{for all } t, u \in \mathbb{R}^+.
\end{align*}
\]
A well-known example of a regular matrix method is the Cesàro method
defined by the matrix $A = \frac{1}{n} I$, where, for any $n \in \mathbb{N}$, $c_{nk} = n^{-1}$ if $k \leq n$ and $c_{nk} = 0$ otherwise. A (trivial) regular method is defined by the unit matrix $I = (i_{nk})$, where $i_{nn} = 1$ and $i_{nk} = 0$ for $n \neq k$. Recall also that a
matrix $A = (a_{nk})$ is called normal if, for any $n \in \mathbb{N}$, $a_{nn} \neq 0$ and $a_{nk} = 0$ if $k > n$. For example, the Cesàro matrix $C_1$ is normal. Every scalar sequence $(c_k)$ defines a diagonal matrix $D(c_k) = (d_{ni})$ by the equalities $d_{kn} = c_n$ and $d_{ni} = 0$ if $n \neq i$. Clearly, a diagonal matrix $D(c_k)$ is regular if and only if $\lim_k c_k = 1$, and it is normal if $c_k \neq 0$ for all $k \in \mathbb{N}$.

Another class of summability methods is determined by sequences $B = (B^i)$ of infinite scalar matrices $B^i = (b^i_{nk})$. Recall (see, for example, [5] and [47]) that a sequence $\xi = (x_k) \in s(\mathcal{X})$ is called $B$-summable to the point $l \in \mathcal{X}$ if $B^i\lim x_k = l$ uniformly in $i$, i.e., if the series $B^i_\mathcal{X}\xi = \sum_k b^i_{nk}x_k$ $(n, i \in \mathbb{N})$ converge in $\mathcal{X}$ and

$$\lim_n |B^n_\mathcal{X}\xi - l| = 0 \text{ uniformly in } i.$$ 

The summability methods $B$ are also known as the sequential matrix methods (SM methods) of summability (see [17], p. 19). In the special case

$$b^i_{nk} = \begin{cases} \frac{1}{n}, & \text{if } i \leq k \leq n + i - 1, \\ 0, & \text{otherwise} \end{cases}$$

the $B$-summability reduces to the so-called almost convergence (see [34]). The almost convergence is a non-matrix method of summability. Any matrix method $B$ can be considered as an SM method $B$ with $B^i = B$ $(i \in \mathbb{N})$. By the unit SM method $\mathcal{I}$ we mean the SM method $B$ with $B^i = \mathcal{I}$ $(i \in \mathbb{N})$.

Let $e^k = (e^k_j)_{j \in \mathbb{N}}$ $(k \in \mathbb{N})$ be the sequences with the elements $e^k_j = 1$ if $j = k$ and $e^k_j = 0$ otherwise. If we define, for an arbitrary sequence $z = (z_k)$, the double sequence $z^{(2)} = (z^{(2)}_k)$ with $z^{(2)}_k = z_k$ $(k, i \in \mathbb{N})$, then every sequence $e^k$ $(k \in \mathbb{N})$ also determines a double sequence $e^{k(2)} = (e^{k}_{ji})_{j, i \in \mathbb{N}}$ such that, for all $i \in \mathbb{N}$, $e^{k}_{ji} = 1$ if $j = k$ and $e^{k}_{ji} = 0$ if $j \neq k$. An F-semi-normed sequence space $(\lambda, g_{\lambda})$ is called an AK-space, if $\lambda$ contains the sequences $e^k$ $(k \in \mathbb{N})$ and for any $u = (u_k) \in \lambda$ we have $\lim_n g_{\lambda}(u - u^{[n]}) = 0$, where $u^{[n]} = \sum_{k=1}^n u_k e^k$. Analogously, an F-semi-normed DS space $(\Lambda, g_{\lambda})$ is called an AK-space (see [42]), if $\Lambda$ contains the sequences $e^{k(2)}$ $(k \in \mathbb{N})$ and for any $u^2 = (u_{ki}) \in \Lambda$ we have $\lim_n g_{\lambda}(u^2 - u^{2[n]}) = 0$, where $u^{2[n]} = \sum_{k=1}^n u_k e^{k(2)}$ with $u_k = (u_{ki})_{i \in \mathbb{N}}$ and $u_k e^{k(2)} = (u_{ki} e^{k}_{ji})_{j, i \in \mathbb{N}}$. Well-known AK-spaces are $\ell^p$ $(p \geq 1)$ with respect to ordinary norms $\|u\|_\infty = \sup_k |u_k|$ and $\|u\|_p = \left(\sum_k |u_k|^p\right)^{1/p}$. It is not difficult to see that $U_0$ and $U_0^p$ $(p \geq 1)$, topologized by norms $\|u\|_\infty = \|\tilde{\mathbf{u}}\|_\infty$ and $\|u\|_\tilde{p} = \|\tilde{\mathbf{u}}\|_\tilde{p}$, are examples of normed DS-AK-spaces.

Let $\Phi = (\phi_k)$ be a sequence of moduli. If $\lambda$ is a solid sequence space, then

$$\lambda(\Phi) = \{u = (u_k) \in s : \Phi(u) = (\phi_k(|u_k|)) \in \lambda\}$$
is also a solid sequence space. Soomer [46] and Kolk [32] proved that if \( \lambda \) is topologized by an absolutely monotone \( F \)-seminorm \( g_\lambda \), i.e., \( g_\lambda (v) \leq g_\lambda (u) \) for all \( u, v \in \lambda \) with \( |v_k| \leq |u_k| \) \((k \in \mathbb{N})\), then \( \lambda(\Phi) \) may be topologized by the absolutely monotone \( F \)-seminorm \( g_{\lambda, \Phi} (u) = g_\lambda (\Phi (u)) \) whenever \( (\lambda, g_\lambda) \) is an \( AK \)-space or the sequence \( \Phi \) satisfies one of two (equivalent) conditions:

1. \( \nu(u) \phi_k (t) \leq B \) \((k \in \mathbb{N}, 0 < u < \delta, t > 0)\) and \( \lim_{u \to 0^+} \nu(u) = 0, \)
2. \( \lim_{u \to 0^+} \sup_{t > 0} \frac{\phi_k (ut)}{\phi_k (t)} = 0. \)

In the following we prove the similar statements about the sets

\[
\lambda (\Phi, B, X) = \left\{ x \in s(X) : \Phi (B x) \right\} = \Lambda (\Phi, B, X),
\]

\[
\Lambda (\Phi, B, X) = \left\{ x \in s(X) : \Phi (B x) \right\} = \Lambda (\Phi, B, X),
\]

\[
\Lambda (\Phi, B, M(X)) = \left\{ x \in s(X) : B x \in M(X) \right\} \cap \Lambda (\Phi, B, X),
\]

where the sequence spaces \( \lambda, \Lambda \) are solid, \( B \) is a matrix method, and \( B \) is an SM-method of summability.

**Theorem 1.** If \( \lambda \) and \( \Lambda \) are solid sequence space, then the sets \( \lambda (\Phi, B, X) \), \( \Lambda (\Phi, B, X) \) and \( \Lambda (\Phi, B, M(X)) \) are GS spaces, i.e., linear subsets of \( s(X) \). Moreover, \( \Lambda (\Phi, B, X) \) and \( \Lambda (\Phi, B, M(X)) \) are solid if

\[
|y_k| \leq |x_k| \implies |B_k^i y| \leq |B_k^i x| \quad (k, i \in \mathbb{N}),
\]

and \( \lambda (\Phi, B, X) \) is solid if

\[
|y_k| \leq |x_k| \implies |B_k y| \leq |B_k x| \quad (k \in \mathbb{N}).
\]

**Proof.** To prove the linearity of the set \( \Lambda (\Phi, B, X) \), fix \( \alpha, \beta \in \mathbb{K} \) and \( x, y \in \Lambda (\Phi, B, X) \). Using the linearity of the operators \( B_k^i \), by axioms (M2) and (M3) we have

\[
\phi_k \left( |B_k^i (\alpha x + \beta y)| \right) \leq \phi_k \left( |\alpha| |B_k^i x| \right) + \phi_k \left( |\beta| |B_k^i y| \right)
\]

\[
\leq (|\alpha| + 1) \phi_k \left( |B_k^i x| \right) + (|\beta| + 1) \phi_k \left( |B_k^i y| \right)
\]

for all \( k, i \in \mathbb{N} \), where \( [c] \) denotes the integer part of a number \( c \in \mathbb{R} \). But this gives \( \alpha x + \beta y \in \Lambda (\Phi, B, X) \) because \( \Lambda \) is linear and solid. The linearity of the subset \( \Lambda (\Phi, B, M(X)) \) of \( \Lambda (\Phi, B, X) \) clearly follows from

\[
\sup_i |B_k^i (\alpha x + \beta y)| \leq |\alpha| \sup_i |B_k^i x| + |\beta| \sup_i |B_k^i y|. \]

Now let \( x \in \Lambda (\Phi, B, X) \) and \( y \in s(X) \) be such that \( |y_k| \leq |x_k| \) \((k \in \mathbb{N})\). Since the moduli \( \phi_k \) are increasing, by (1) we get

\[
\phi_k \left( |B_k^i y| \right) \leq \phi_k \left( |B_k^i x| \right) \quad (k, i \in \mathbb{N}),
\]

(3)
and in view of solidity of $\Lambda$, the sequence $\Phi(By)$ is in $\Lambda$. Thus $y \in \Lambda(\Phi, B, X)$. Hence, $\Lambda(\Phi, B, X)$ is solid if (1) holds. The solidity of $x, y \in \Lambda(\Phi, B, M(X))$ is obvious.

The statements about the set $\lambda(\Phi, B, X)$ follow similarly, with $B_k$ instead of $B_k'$.

An F-seminorm $g_\Lambda$ on a DS space $\Lambda$ is said to be absolutely monotone if $g_\Lambda(v^2) \leq g_\Lambda(u^2)$ for all $u^2, v^2 \in \Lambda$ with $|v_k| \leq |u_k|$ ($k, i \in \mathbb{N}$).

**Theorem 2.** Let $\Lambda$ be a solid DS space which is topologized by an absolutely monotone F-seminorm $g_\Lambda$.

a) If a sequence of moduli $\Phi = (\phi_k)$ satisfies one of two (equivalent) conditions (M5) and (M6), then the GS space $\Lambda(\Phi, B, X)$ may be topologized by the F-seminorm

$$g_{\Lambda,B}(x) = g_\Lambda(\Phi(Bx)).$$

Moreover, if $g_\Lambda$ is an F-norm on $\Lambda$, the space $X$ is normed, and $B$ satisfies the condition

$$Bx = 0 \implies x = 0,$$

then $g_{\Lambda,B}$ is an F-norm on $\Lambda(\Phi, B, X)$. The F-seminorm (or F-norm) $g_{\Lambda,B}$ is absolutely monotone if (1) holds.

b) If $(\Lambda, g_\Lambda)$ is an AK-space, then the GS space $\Lambda(\Phi, B, M(X))$ may be topologized by the F-seminorm $g_{\Lambda,B}$ for an arbitrary sequence of moduli $\Phi$.

Moreover, if $g_\Lambda$ is an F-norm in $\Lambda$, the space $X$ is normed, and $B$ satisfies (4), then $g_{\Lambda,B}$ is an F-norm on $\Lambda(\Phi, B, M(X))$. The F-seminorm (or F-norm) $g_{\Lambda,B}$ is absolutely monotone on GS space $\Lambda(\Phi, B, M(X))$ whenever $B$ satisfies (1).

**Proof.** a) First, we prove that $g_{\Lambda,B}$ is an F-seminorm. Since $g_\Lambda$ is an F-seminorm, (N1) holds by (M1). Because the operator $B$ is linear, axiom (N2) follows immediately from the subadditivity of $\phi_k$ and $g_\Lambda$. If $|\alpha| \leq 1$, then by (M3) we get

$$\phi_k(|B_k^i(\alpha x)|) = \phi_k(|\alpha||B_k^i x|) \leq \phi_k(|B_k^i x|) \quad (k, i \in \mathbb{N}).$$

Since $g_\Lambda$ is absolutely monotone,

$$g_{\Lambda,B}(\alpha x) = g_\Lambda\left(\left(\phi_k\left(|B_k^i(\alpha x)|\right)\right)\right) \leq g_\Lambda\left(\left(\phi_k\left(|B_k^i x|\right)\right)\right) = g_{\Lambda,B}(x),$$

i.e., (N3) is true.

To prove (N4), let $x \in \Lambda(\Phi, B, X)$. Using the equivalence of (M5) and (M6) (see [32], Remark 1), we may assume that $\Phi$ satisfies (M5). Therefore, if $\lim_n \alpha_n = 0$ ($\alpha_n \in \mathbb{K}$), we can fix an index $n_0$ such that $|\alpha_n| < \delta$ for all $n \geq n_0$. Then by (M5) we obtain

$$\phi_k\left(|B_k^i(\alpha_n x)|\right) \leq \nu(|\alpha_n|) \phi_k\left(|B_k^i x|\right)$$
for all \( k, i \in \mathbb{N} \). So, since \( g_\Lambda \) is absolutely monotone, we get
\[
g_\Lambda \left( \Phi \left( B (\alpha_n x) \right) \right) \leq g_\Lambda \left( \nu \left( |\alpha_n| \right) \Phi \left( B x \right) \right) \quad (n \geq n_0).
\]
But this yields \( \lim_n g_{\Lambda,B} (\alpha_n x) = 0 \) because \( \lim_n \nu (|\alpha_n|) = 0 \). Thus (N4) holds and \( g_{\Lambda,B} \) is an F-seminorm on \( \Lambda (\Phi, B, X) \).

Let \( g_\Lambda \) be an F-norm and let \((X, \| \cdot \|_X)\) be a normed space. If \( g_{\Lambda,B} (x) = 0 \), then, using also (M1), we have
\[
\| B_i^k x \|_X = 0 \quad (k, i \in \mathbb{N}),
\]
which gives \( x = 0 \) by (4). So, \( g_{\Lambda,B} \) is an F-norm on \( \Lambda (\Phi, B, X) \) in this case.

Now, suppose that (1) is satisfied. Then (3) holds, and since \( g_\Lambda \) is absolutely monotone,
\[
g_{\Lambda,B} (y) = g_\Lambda \left( \left( \phi_k \left( |B_i^k y| \right) \right) \right) \leq g_\Lambda \left( \left( \phi_k \left( |B_i^k x| \right) \right) \right) = g_{\Lambda,B} (x).
\]
Consequently, F-seminorm (or F-norm) \( g_{\Lambda,B} \) is absolutely monotone if (1) holds.

b) By the proof of a) it suffices to show that the functional
\[
g_{\Lambda,B} : \Lambda (\Phi, B, \mathcal{M}(X)) \to \mathbb{K}
\]
satisfies axiom (N4). Let \( \lim_n \alpha_n = 0 \) and let \( x \) be an arbitrary element from \( \Lambda (\Phi, B, \mathcal{M}(X)) \). Then \( \Phi (B x) \in \Lambda \), and since \( \Lambda \) is an AK-space,
\[
\lim_n g_\Lambda \left( \Phi (B x) - \Phi (B x)^{[n]} \right) = 0. \tag{5}
\]
Using the equality
\[
\Phi (B x) - \Phi (B x)^{[n]} = \Phi \left( B x - (B x)^{[n]} \right),
\]
by (5) we can find, for fixed \( \varepsilon > 0 \), an index \( m \) such that
\[
g_\Lambda \left( \Phi \left( B x - (B x)^{[m]} \right) \right) < \varepsilon / 2. \tag{6}
\]

The double sequence \( B x \in \mathcal{M}(X) \) determines the single sequence \( (\tilde{z}_k) \) by \( \tilde{z}_k = \sup_i |B_i^k x| \quad (k \in \mathbb{N}) \). Since
\[
\lim_n \phi_k (|\alpha_n \tilde{z}_k|) = 0 \quad (k \in \mathbb{N})
\]
and \( g_\Lambda \) satisfies (N4), we have that
\[
\lim_n g_\Lambda \left( \phi_k (|\alpha_n \tilde{z}_k|) e^{k(2)} \right) = 0 \quad (k \in \mathbb{N}). \tag{7}
\]
Further, since \( g_\Lambda \) satisfies (N2) and is absolutely monotone, we may write
\[
g_\Lambda \left( \Phi \left( B (\alpha_n x) \right)^{[m]} \right) = g_\Lambda \left( \sum_{k=1}^{m} \phi_k \left( |\alpha_n B_i^k x| \right) e^{k(2)} \right).
\]
\[ \leq \sum_{k=1}^{m} g_{\Lambda} \left( \left( \phi_{k} \left( |\alpha_{n} B_{k}^{i} x| \right) \right) \right) e^{k(2)} \]
\[ \leq \sum_{k=1}^{m} g_{\Lambda} \left( \phi_{k} \left( |\alpha_{n} \tilde{z}_{k}| \right) e^{k(2)} \right). \]

This yields
\[ \lim_{n} g_{\Lambda} \left( \Phi \left( B (\alpha_{n} x) \right) \right) = 0 \]
because of (7). Thus there exists an index \( n_{0} \) such that, for all \( n \geq n_{0} \),
\[ |\alpha_{n}| \leq 1 \quad \text{and} \quad g_{\Lambda} \left( \Phi \left( |\alpha_{n} (Bx)^{[m]} | \right) \right) < \varepsilon/2. \]  
(8)

Now, by (6) and (8) we get
\[ g_{\Lambda,B} \left( \alpha_{n} x \right) = g_{\Lambda} \left( \Phi \left( B (\alpha_{n} x) \right) \right) \]
\[ \leq g_{\Lambda} \left( \Phi \left( |\alpha_{n}| (Bx - (Bx)^{[m]}) \right) \right) + g_{\Lambda} \left( \Phi \left( |\alpha_{n}| (Bx)^{[m]} \right) \right) \]
\[ \leq g_{\Lambda} \left( \Phi \left( Bx - (Bx)^{[m]} \right) \right) + g_{\Lambda} \left( \Phi \left( |\alpha_{n}| (Bx)^{[m]} \right) \right) \]
\[ < \varepsilon/2 + \varepsilon/2 = \varepsilon \]
for \( n \geq n_{0} \). Hence, \( \lim_{n} g_{\Lambda,B} \left( \alpha_{n} x \right) = 0 \), i.e., (N4) is true for \( g_{\Lambda,B} \).

Let \( \lambda \subset s \) be a solid sequence space and let \( B = (b_{kj}) \) be an infinite scalar matrix. Denoting by \( \lambda^{2} \) the set of all double sequences \( x^{(2)} \) with \( x \in \lambda \), and using the sequence \( B = (B^{i}) \) of matrices \( B^{i} = (b_{kj}^{i}) \) with the elements \( b_{kj}^{i} = b_{kj} \) (\( i \in \mathbb{N} \)) it is easy to see that \( \lambda (\Phi, B, X) \) is isomorphic to the space \( \lambda^{2} (\Phi, B, X) \) of type \( \lambda (\Phi, B, X) \). In addition, if \( \lambda \) is topologized by an (absolutely monotone) F-seminorm \( g_{\lambda} \), then the equality
\[ g_{\lambda,B} \left( x^{(2)} \right) = g_{\lambda} \left( x \right) \]
defines an (absolutely monotone) F-seminorm on \( \lambda^{2} (\Phi, B, X) \). Thus, since \( x \in \mathcal{M}(X) \) for every \( x \in \lambda^{2} (\Phi, B, X) \) and \( (\lambda^{2} (\Phi, B, X), g_{\lambda,B}) \) is an AK-space if and only if \( (\lambda (\Phi, B, X), g_{\lambda}) \) is, Theorem 2 gives the following topologization theorem for \( \lambda (\Phi, B, X) \).

**Theorem 3.** Let \( \lambda \) be a solid sequence space topologized by an absolutely monotone F-seminorm \( g_{\lambda} \).

a) If a sequence of moduli \( \Phi = (\phi_{k}) \) satisfies one of two (equivalent) conditions (M5) and (M6), then the GS space \( \lambda (\Phi, B, X) \) may be topologized by the F-seminorm
\[ g_{\lambda,B} \left( x \right) = g \left( \lambda (Bx) \right). \]
Moreover, if $g_\lambda$ is an F-norm on $\lambda$, the space $X$ is normed, and $B$ satisfies the condition

$$Bx = 0 \implies x = 0,$$

(9)

then $g_{\lambda,B}$ is an F-norm on $\lambda(\Phi, B, X)$. The F-seminorm (or F-norm) $g_{\lambda,B}$ is absolutely monotone if (2) holds.

b) If $(\lambda, g_\lambda)$ is an AK-space, then the GS space $\lambda(\Phi, X)$ may be topologized by the F-seminorm $g_{\lambda,B}$ for an arbitrary sequence of moduli $\Phi$. Moreover, if $g_\lambda$ is an F-norm, the space $X$ is normed, and $B$ satisfies (9), then $g_{\lambda,B}$ is an F-norm on $\lambda(\Phi, B, X)$. The F-seminorm (or F-norm) $g_{\lambda,B}$ is absolutely monotone whenever $B$ satisfies (2).

**Remark 1.** It is not difficult to see that in Theorems 1–3 we may write $X$ instead of $X$ whenever the matrices $B_i$ ($i \in \mathbb{N}$) and $B$ are diagonal or, more generally, whenever each row of these matrices contains not more than one non-zero element.

**Remark 2.** Ghosh and Srivastava [27] considered, for one modulus $\phi$ and for a sequence $X$ of Banach spaces $(X_k, \|\cdot\|_k)$ ($k \in \mathbb{N}$), the GS space

$$\lambda(\phi, X) = \{x : \phi(x) = (\phi(||x||_k)) \in \lambda\},$$

where $\lambda$ is a solid sequence space. They assert (see [27], Theorem 3.1) that if $\lambda$ is topologized by an absolutely monotone paranorm $g$, then

$$g_{\phi}(x) = g(\phi(x))$$

is a paranorm on $\lambda(\phi, X)$. But this is not true in general. Indeed, if $\phi$ is a bounded modulus and the solid sequence space $\ell_\infty$ is topologized by the absolutely monotone norm $g(u) = \sup_k |u_k|$, then $\ell_\infty(\phi, X) = s(X)$, and so, $\ell_\infty(\phi, X)$ contains an unbounded sequence $z = (z_k)$. If now $(z_{k_i})$ is a subsequence of $z$ such that $z_{k_i} \neq 0$ and $\lim_i \|z_{k_i}\| = \infty$, then, defining

$$\alpha_n = \begin{cases} (\|z_{k_i}\|_{k_i})^{-1}, & \text{if } n = k_i \ (i \in \mathbb{N}), \\ 0, & \text{otherwise}, \end{cases}$$

we get the sequence $(\alpha_n)$ with $\lim_n \alpha_n = 0$. Since

$$\phi(\|\alpha_{k_i}z_{k_i}\|_{k_i}) = \phi(1) > 0 \ (i \in \mathbb{N}),$$

we have that

$$\lim_n g_{\phi}(\alpha_n z) = \lim_{n} \sup_k \|\alpha_n z_k\| \neq 0.$$

Thus $g_{\phi}$ does not satisfy axiom (N4) and, consequently, is not a paranorm on $\ell_\infty(\phi, X)$ if the modulus $\phi$ is bounded. Theorem 3 a) (for $B = I$) and Remark 1 show that if the solid sequence space $\lambda$ is topologized by an absolutely monotone F-seminorm (or a paranorm with (N3)) $g$, then $g_{\phi}$ is an absolutely monotone F-seminorm (paranorm) on the GS space $\lambda(\phi, X)$ whenever $(\lambda, g)$
is an AK-space or the modulus $\phi$ satisfies one of the following (equivalent) conditions:

(M5) there exists a function $\nu$ and a number $\delta > 0$ such that $\phi(ut) \leq \nu(u)\phi(t)$ ($0 \leq u < \delta$, $t \geq 0$) and $\lim_{u \to 0^+} \nu(u) = 0$,

(M6) $\lim_{u \to 0^+} \sup_{t > 0} \frac{\phi(ut)}{\phi(t)} = 0$.

These conditions clearly fail if $\phi$ is bounded, since by $\sup_{t > 0} \phi(t) = M < \infty$ we have

$$\sup_{t > 0} \frac{\phi(ut)}{\phi(t)} \geq M^{-1} \sup_{t > 0} \phi(ut) = 1$$

for any fixed $u > 0$.

3. Applications related to strong summability domains

Let $A = (a_{nk})$ be a non-negative matrix, i.e., $a_{nk} \geq 0$ ($n,k \in \mathbb{N}$). We say that $A$ is column-positive if for any $k \in \mathbb{N}$ there exists an index $n_k$ such that $a_{n_k,k} > 0$. Obviously, any normal non-negative matrix is column-positive, and a diagonal matrix $D(c_k)$ is column-positive if $c_k > 0$ for all $k \in \mathbb{N}$. A sequence $u = (u_k) \in s$ is called strongly $A$-summable with index $p \geq 1$ if $\lim_{n} \sum_{k} a_{nk}|u_k|^p = 0$, and strongly $A$-bounded with index $p$ if $\sup_{n} \sum_{k} a_{nk}|u_k|^p < \infty$. It is clear that the set $c_0^p[A]$ of all strongly $A$-summable with index $p$ sequences and the set $\ell^p[A]$ of all strongly $A$-bounded with index $p$ sequences are solid linear spaces and $c_0^p[A] \subset \ell^p[A]$.

Moreover, the functional

$$g^p_{[A]}(u) = \sup_{n} \left( \sum_{k} a_{nk}|u_k|^p \right)^{1/p}$$

is a seminorm on $\ell^p[A]$ and $c_0^p[A]$, and it is a norm if $A$ is column-positive.

Natural generalizations of sequence spaces $c_0^p[A]$ and $\ell^p[A]$ are related to arbitrary solid F-seminormed sequence spaces $(\lambda, g_\lambda)$ and $(\Lambda, g_\Lambda)$. It is easy to see that the sets

$$\lambda^p[A] = \left\{ u \in s : A^{1/p} (|u|^p) = \left( \sum_{k} a_{nk}|u_k|^p \right)^{1/p} \in \lambda \right\},$$

$$\Lambda^p[A] = \left\{ u^2 \in s^2 : A^{1/p} (|u^2|^p) = \left( \sum_{k} a_{nk}|u_k|^p \right)^{1/p} \in \Lambda \right\}$$

are solid linear subspaces of $s$ and $s^2$, respectively. In addition, if F-seminorms $g_\lambda$ and $g_\Lambda$ are absolutely monotone, then the functionals

$$g^p_{\lambda,[A]}(u) = g_\lambda \left( A^{1/p} (|u|^p) \right) \quad \text{and} \quad g^p_{\Lambda,[A]}(u^2) = g_\Lambda \left( A^{1/p} (|u^2|^p) \right)$$
define F-seminorms, respectively, on $\lambda^p[A]$ and $\Lambda^p[A]$. Moreover, if $A$ is column-positive, then $g^p_{\lambda,[A]}$ (or $g^p_{\Lambda,[A]}$) is an F-norm (a norm) whenever the space $\lambda$ (or $\Lambda$) is F-normed (normed).

As a special case of $\Lambda^p[A]$ we will consider the DS space

$$U\lambda^p[A] = \{ u^2 \in s^2 : A^{1/p} (|u^2|^p) \in U\lambda \},$$

which may be topologized by the F-seminorm

$$g^p_{\lambda,[A]}(u) = g^p_{\lambda} \left( \tilde{A}^{1/p} (|u^2|^p) \right)$$

if a solid sequence space $\lambda$ is topologized by an absolutely monotone F-seminorm $g_{\lambda}$ and

$$\tilde{A}^{1/p} (|u^2|^p) = \left( \sup_n \left( \left( \sum_k a_{nk} |u_{ki}|^p \right)^{1/p} \right) \right)_{n \in \mathbb{N}}.$$

Let $p = (p_k)$ be a bounded sequence of positive numbers with $r = \max \{1, \sup_k p_k\}$, let $B = (b_{nk})$ be an infinite scalar matrix, and let $B$ be an SM method. For a sequence of moduli $\Phi = (\phi_k)$ and solid sequence spaces $\lambda \subset s$, $\Lambda \subset s^2$, we consider, as some generalizations of $\lambda^p[A]$ and $\Lambda^p[A]$, the sets

$$\lambda[A^{1/r}, B, \Phi, p, X] = \{ x \in s(X) : A^{1/r} (\Phi^p(Bx)) \in \lambda \},$$

$$\Lambda[A^{1/r}, B, \Phi, p, X] = \{ x \in s(X) : A^{1/r} (\Phi^p(Bx)) \in \Lambda \},$$

where,

$$A^{1/r} (\Phi^p(Bx)) = \left( \left( \sum_k a_{nk} \left( \phi_k \left( \left| \sum_j b_{kj} x_j \right| \right) \right) \right)^{p_k} \right)_{n \in \mathbb{N}}^{1/r},$$

$$A^{1/r} (\Phi^p(Bx)) = \left( \left( \sum_k a_{nk} \left( \phi_k \left( \left| \sum_j b_{ij} x_j \right| \right) \right) \right)^{p_k} \right)_{n,i \in \mathbb{N}}^{1/r}.$$
Using the equalities \( p_k = (p_k/r)r \ (k \in \mathbb{N}) \) and denoting by \( \Phi^{p/r} \) the sequence of moduli \( \phi_k^{p/r}(t) = (\phi_k(t))^{p_k/r} \ (t \in \mathbb{R}^+, \ k \in \mathbb{N}) \), we may write

\[
\lambda[A^{1/r}, B, \Phi, p, X] = \left\{ x \in s_B(X) : \Phi^{p/r}(Bx) \in \lambda^r[A] \right\}
= \lambda^r[A] \left( \Phi^{p/r}, B, X \right),
\]

(10)

\[
\Lambda[A^{1/r}, B, \Phi, p, X] = \left\{ x \in s_B(X) : \Phi^{p/r}(Bx) \in \Lambda^r[A] \right\}
= \Lambda^r[A] \left( \Phi^{p/r}, B, X \right).
\]

(11)

Thus, since the spaces \( \lambda^r[A] \) and \( \Lambda^r[A] \) are solid, Theorem 1 shows that \( \lambda[A^{1/r}, B, \Phi, p, X] \) and \( \Lambda[A^{1/r}, B, \Phi, p, X] \) are GS spaces. Remark 1 shows that we get the GS spaces \( \lambda[A^{1/r}, B, \Phi, p, X] \) and \( \Lambda[A^{1/r}, B, \Phi, p, X] \), for example, in the special case, when \( B \) is a diagonal matrix and \( \Phi \) is a sequence of diagonal matrices.

The representations (10) and (11) are also useful for the topologization of sequence spaces \( \lambda[A^{1/r}, B, \Phi, p, X] \) and \( \Lambda[A^{1/r}, B, \Phi, p, X] \). But first of all, we prove an auxiliary result about the property AK of the spaces \( \lambda[A^{1/r}, B, \Phi, p, X] \) and \( \Lambda[A^{1/r}, B, \Phi, p, X] \).

**Lemma 1.** Let \( p \geq 1 \) and let \( A = (a_{nk}) \) be a non-negative infinite matrix. Suppose that \( \lambda \subset s \) is a solid AK-space with respect to an absolutely monotone \( F \)-seminorm \( g_\lambda \).

(i) If

\[
a_k = (((a_{nk})^{1/p})_{n \in \mathbb{N}} \in \lambda \ (k \in \mathbb{N}),
\]

(12)

then \( (\lambda^p[A], g^p_{\lambda,A}) \) is an AK-space.

(ii) If the matrix \( A \) is row-finite (i.e., for any \( n \in \mathbb{N} \) there exists an index \( k_n \) with \( a_{nk} = 0 \ (k > k_n) \)), and (12) holds, then \( (U \lambda^p[A], g^p_{\lambda,A}) \) is an AK-space.

**Proof.** The proof of statement (i) is quite similar to the proof of Lemma 1 from [33] and therefore it is omitted.

To prove (ii), let \( u^2 \in U \lambda^p[A] \). Thus \( \tilde{A}^{1/p} ([u^2]^p) \in \lambda \), and since \( (\lambda, g_\lambda) \) is an AK-space,

\[
\lim_m g_\lambda \left( \tilde{A}^{1/p} ([u^2]^p) - \tilde{A}^{1/p} ([u^2]^p)^{[mq]} \right)
= \lim_m g_\lambda \left( \left[ 0, \ldots, 0, \sup_i \left( \sum_k a_{m+1,k}|u_{ki}|^p \right)^{1/p} \right] \right) = 0.
\]

(13)

By condition (12) and by

\[
\tilde{A}^{1/p} ([e^{(2)}]^p) = \left( \sup_i (a_{nj})^{1/p} \right)_{n \in \mathbb{N}} = a_j \ (j \in \mathbb{N})
\]
we conclude that $U\lambda^p[A]$ contains the sequences $e^{j(2)}$. To prove the equality $\lim_m u^{[m]} = u$ in $U\lambda^p[A]$, we use the inequality
\[
g_{\lambda,|A|} (u^2 - u^{2[m]}) \leq \sum_{n=1}^{s} g_{\lambda} \left( \sup_{i} \left( \sum_{k=m+1}^{\infty} a_{nk} |u_{ki}|^p \right)^{1/p} \right) e^n \]
\[
+ g_{\lambda} \left( 0, \ldots, 0, \sup_{i} \left( \sum_{k=m+1}^{\infty} a_{s+1,k} |u_{ki}|^p \right)^{1/p}, \ldots \right) \]
\[
= G_{sm}^1 + G_{sm}^2.
\]
Let $\varepsilon > 0$. As $g_{\lambda}$ is absolutely monotone, we have
\[
G_{m_{0},m_{0}}^2 \leq g_{\lambda} \left( \tilde{A}^{1/p} (|u|^p) - \tilde{A}^{1/p} (|u|^p)^{[m]} \right),
\]
and by (13) we get $\lim_m G_{m_{0},m_{0}}^2 = 0$. Thus, there exists a number $m_{0} \in \mathbb{N}$ with
\[
G_{m_{0},m_{0}}^2 < \varepsilon.
\]
Since the matrix $A$ is row-finite, we can find $m_{1} \geq m_{0}$ such that for all $n = 1, 2, \ldots, m_{0}$ and $i \in \mathbb{N}$ one has
\[
\sum_{k=m+1}^{\infty} a_{nk} |u_{ki}|^p = 0 \quad (m \geq m_{1}),
\]
which yields
\[
G_{m_{0},m_{0}}^1 = 0 \quad (m \geq m_{1}).
\]
Hence, using the inequalities $G_{m_{0},m_{0}}^2 \leq G_{m_{0},m_{0}}^2$ ($m \geq m_{0}$), we have that
\[
g_{\lambda,|A|} (u^2 - u^{2[m]}) \leq G_{m_{0},m_{0}}^1 + G_{m_{0},m_{0}}^2 < 0 + \varepsilon = \varepsilon
\]
if $m \geq m_{1}$. Consequently, $\lim_m u^{[m]} = u$ in $U\lambda^p[A]$. The proof is completed. \hfill \Box

Now we can determine F-seminorms on GS spaces $\lambda[A^{1/r}, B, \Phi, p, X]$ and $\Lambda[A^{1/r}, B, \Phi, p, X]$.

**Proposition 1.** Let $\Phi = (\phi_k)$ be a sequence of moduli and let $p = (p_k)$ be a bounded sequence of positive numbers and $r = \max\{1, \sup_k p_k\}$. Let $A = (a_{nk})$ be a non-negative infinite matrix and let $B = (b_{nk})$ be an infinite matrix of scalars. Suppose that $(X_k, |\cdot|_k)$ is a seminormed space, $X$ is a sequence of seminormed spaces $(X_k, |\cdot|_k) \ (k \in \mathbb{N})$, and $\lambda \subset s$ is a solid sequence space topologized by an absolutely monotone F-seminorm $g_{\lambda}$. 

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a) If the sequence of moduli $\Phi^p/r$ satisfies one of conditions (M5) and (M6), then the GS space $\Lambda[A^{1/r}, B, \Phi, p, X]$ may be topologized by the F-seminorm

$$g_{\Lambda,A,B}^\Phi(x) = g_\lambda \left( A^{1/r} \left( \Phi^p(Bx) \right) \right).$$

b) If $(\lambda, g_\lambda)$ is an AK-space and condition (12) holds with $p = r$, then $g_{\Lambda,A,B}^\Phi$ is an F-seminorm on $\Lambda[A^{1/r}, B, \Phi, p, X]$ for an arbitrary sequence of moduli $\Phi$.

If, in a) and b), $g_\lambda$ is an F-norm, $X$ is normed, $A$ is column-positive, and (9) holds, then $g_{\Lambda,A,B}^\Phi$ is an F-norm on $\Lambda[1/r, B, \Phi, p, X]$. Moreover, for a diagonal matrix $B = D(c_k)$, we get the absolutely monotone F-seminorm (or F-norm) $g_{\Lambda,A,D(c_k)}^\Phi$ on the GS space $\Lambda[1/r, D(c_k), \Phi, p, X]$.

**Proof.** Statement a) follows from (10) and Theorem 3 a) because

$$g_{\Lambda,A,B}^\Phi(x) = g_\lambda \left( A^{1/r} \left( \left( \Phi^p(Bx) \right) \right) \right) = g_\lambda \left( A^{1/r} \left( \Phi^p(Bx) \right) \right)$$

for any $x \in X[1/r, \Phi^p/B, X]$. Analogously, we deduce statement b) from (11) and Theorem 3 b) in view of Lemma 1(i).

Let us investigate the topologization of spaces of type $\Lambda[A^{1/r}, B, \Phi, p, X]$ in the case $\Lambda = U\Lambda$.

**Proposition 2.** Let $\Phi, p, A, X$, and $B$ be the same as in Proposition 1 and let $B$ be a sequence of infinite matrices $B^i = (b_{nk})$.

a) If $\Phi^p/r$ satisfies one of conditions (M5) and (M6), then the GS space $U\Lambda[A^{1/r}, B, \Phi, p, X]$ may be topologized by the F-seminorm

$$g_{U\Lambda,A,B}^\Phi(x) = g_\lambda \left( A^{1/r} \left( \Phi^p(Bx) \right) \right)$$

$$= g_\lambda \left( \sup \left( \sum_k a_{nk} \left( \phi_k \left( \left| \sum_j b_{kj}x_j \right| \right) \right) \right) \right)_{\lambda \in N}.$$

b) Suppose that $(\lambda, g_\lambda)$ is an AK-space and the moduli $\phi_k$ ($k \in N$) are unbounded. If the matrix $A$ is row-finite and column-positive, and (12) holds with $p = r$, then $g_{U\Lambda,A,B}^\Phi$ is an F-seminorm on $U\Lambda[A^{1/r}, B, \Phi, p, X]$.

If, in a) and b), $g_\lambda$ is an F-norm, $X$ is normed, and (4) is true, then $g_{U\Lambda,A,B}^\Phi$ is an F-norm on $U\Lambda[A^{1/r}, B, \Phi, p, X]$. Moreover, $g_{U\Lambda,A,B}^\Phi$ is an absolutely monotone F-seminorm (or F-norm) on the GS space $U\Lambda[A^{1/r}, B, \Phi, p, X]$ if $B$ is a sequence of diagonal matrices.

**Proof.** Statement a) follows from Theorem 2 a) in view of (11).
b) Under our assumptions, the space \( U_{\lambda}[A, g^*, \lambda, [A]] \) has the property \( AK \) by Lemma 1 (ii). In addition, since our non-negative matrix \( A \) is column-positive, for any fixed \( k \) there exists an index \( n_k \) such that \( a_{nk,k} > 0 \). Thus, using the inequality

\[
\left( \left( \phi_k \left( \sum_j b_{kj} x_j \right) \right)^p \right)^{1/r} \leq \left( a_{nk,k} \right)^{-1/r} \sup_i \left( \sum_k a_{nk} \left( \phi_k \left( \sum_j b_{kj} x_j \right) \right)^p \right)^{1/r},
\]

by \( A^{1/r} (\Phi^p(Bx)) \in \mathcal{M} \) we get

\[
\sup_i \left( \left( \phi_k \left( \sum_j b_{kj} x_j \right) \right)^p \right)^{1/r} < \infty.
\]

But this yields

\[
\sup_i \left| \sum_j b_{kj} x_j \right| < \infty \quad (k \in \mathbb{N})
\]

because the moduli \( \phi_k^{p/r}(t) = (\phi_k(t))^{p/k} \) \( (k \in \mathbb{N}) \) are unbounded. Consequently, for any \( x \in U_{\lambda}[A^{1/r}, \mathcal{B}, \Phi, p, X] \) we have \( Bx \in \mathcal{M}(X) \). Hence, by equality (11) we have

\[
U_{\lambda}[A^{1/r}, \mathcal{B}, \Phi, p, X] = U_{\lambda^r}[A] \left( \Phi^{p/r}, \mathcal{B}, \mathcal{M} (X) \right),
\]

and b) follows from Theorem 2 b). \( \square \)

4. Some special cases

In the following we apply Propositions 1 and 2 for the topologization of GS spaces

\[
\lambda[A, B, \Phi, p, X] = \{ x \in s(X) : A (\Phi^p(Bx)) \in \lambda \},
\]

\[
U_{\lambda}[A, \mathcal{B}, \Phi, p, X] = \{ x \in s(X) : A (\Phi^p(Bx)) \in U\lambda \},
\]

where

\[
A(\Phi^p(Bx)) = \left( \sum_k a_{nk} \left( \phi_k \left( \sum_j b_{kj} x_j \right) \right)^p \right)_{n \in \mathbb{N}}
\]

\[
A(\Phi^p(Bx)) = \left( \sum_k a_{nk} \left( \phi_k \left( \sum_j b_{kj} x_j \right) \right)^p \right)_{n, i \in \mathbb{N}}
\]
Corollary 1. Let $A$, $B$, $X$, $X$, and $\lambda$ be the same as in Proposition 1.

a) If the sequence of moduli $\Phi$ satisfies one of conditions (M5) and (M6), then the GS space

$$\lambda[A, B, \Phi, X] = \{x \in s(X) : A(\Phi(Bx)) \in \lambda\}$$

may be topologized by the F-seminorm

$$g_{\lambda,A,B}^\Phi(x) = g_\lambda(A(\Phi(Bx)))$$

$$= g_\lambda\left(\left(\sum_k a_{nk} \left(\phi_k \left(\sum_j b_{kj}x_j\right)\right)\right)\right).$$

b) If $(\lambda, g_\lambda)$ is an AK-space and condition (12) holds with $p = 1$, then $g_{\lambda,A,B}^\Phi$ is an F-seminorm on $\lambda[A, B, \Phi, X]$ for an arbitrary sequence of moduli $\Phi$.

If, in a) and b), $g_\lambda$ is an F-norm, $X$ is normed, $A$ is column-positive, and $B$ satisfies (9), then $g_{\lambda,A,B}^\Phi$ is an F-norm on $\lambda[A, B, \Phi, X]$. Moreover, $g_{\lambda,A,B}^\Phi$ is an absolutely monotone F-seminorm (or F-norm) on the GS space $\lambda[A, B, \Phi, X]$ if $B$ is a diagonal matrix.

Corollary 2. Let $\Phi$, $A$, $X$, $X$, and $B$ be the same as in Proposition 2.

a) If $\Phi$ satisfies one of conditions (M5) and (M6), then the sequence space $U\lambda[A, B, \Phi, X]$ may be topologized by the F-seminorm

$$g_{U\lambda,A,B}^\Phi(x) = g_\lambda\left(\tilde{A}(\Phi(Bx))\right)$$

$$= g_\lambda\left(\left(\sup_i\sum_k a_{nk} \left(\phi_k \left(\sum_j b_{kj}x_j\right)\right)\right)\right).$$

b) Suppose that $(\lambda, g_\lambda)$ is an AK-space and the moduli $\phi_k$ ($k \in \mathbb{N}$) are unbounded. If the matrix $A$ is row-finite and column-positive, and (12) holds with $p = 1$, then $g_{U\lambda,A,B}^\Phi$ is an F-seminorm on $U\lambda[A^{1/r}, B, \Phi, X]$.

If, in a) and b), $g_\lambda$ is an F-norm, $X$ is normed, and $B$ satisfies (4), then $g_{U\lambda,A,B}^\Phi$ is an F-norm on $U\lambda[A^{1/r}, B, \Phi, X]$. Moreover, $g_{U\lambda,A,B}^\Phi$ is an absolutely monotone F-seminorm (or F-norm) on the GS space $U\lambda[A^{1/r}, B, \Phi, X]$ if $B$ is a sequence of diagonal matrices.

First investigations of spaces of type $\lambda[A, B, \Phi, X]$ are related to the case $B = I$ and $\phi_k = \phi$ ($k \in \mathbb{N}$). Ruckle [44] considered the space

$$\ell[I, \phi] = \{u \in s : \sum_k \phi(|u_k|) < \infty\}$$
and Maddox [35] introduced the sequence spaces \( c_0[C_1, \phi] \) and \( \ell_\infty[C_1, \phi] \). The spaces \( \lambda[A, \phi] \) and \( \lambda[C_1, \phi, X] \) (\( X \) is a Banach space) are studied, respectively, in [10] and [11]. Corollary 1 allows to determine F-seminorm topologies for sequence spaces from [15] and [20]. It also extends Theorem 2.6 of [9], which determines the paranorm \( g_\infty^{\phi} ) on \( \lambda[D(k^{-s}), I, \phi, X] \) if \( s > 0 \) and \( \lambda \) is a Banach space with the property AK.

Further corollaries of Propositions 1 and 2 deal with the sequence \( p \) and are related to \( \lambda \in \{ \ell_\infty, c_0, \ell_r \} \). It is clear that \( \ell_\infty \) and \( c_0 \) are solid sequence spaces with the absolutely monotone norm \( \|u\|_\infty = \sup_k |u_k| \). Since, moreover, \((c_0, \| \cdot \|_\infty)\) is an AK-space, and for \( \lambda \in \{ \ell_\infty, c_0 \} \) we have

\[
(|u_k|) \in \lambda \iff (|u_k|^q) \in \lambda \quad (q > 0),
\]

Proposition 1 immediately yields the following corollary.

**Corollary 3.** Let \( \Phi, p, A, X, X, \) and \( B \) be the same as in Proposition 1.

a) If the sequence of moduli \( \Phi^{p/r} \) satisfies one of conditions (M5) and (M6), then the GS space \( \ell_\infty[A, B, \Phi, p, X] \) may be topologized by the F-seminorm

\[
g_{\infty, A, B}^{\Phi, p} (x) = \sup_n (A(\Phi^p(Bx))) = \sup_n \left( \sum_k a_{nk} \left( \phi_k \left( \sum_j b_{kj} x_j \right) \right)^p \right)^{1/r}.
\]

b) If the matrix \( A \) is such that

\[
\lim_n a_{nk} = 0 \quad (k \in \mathbb{N}),
\]

then \( g_{\infty, A, B}^{\Phi, p} \) is an F-seminorm on \( c_0[A, B, \Phi, p, X] \) for an arbitrary sequence of moduli \( \Phi \).

If, in a) and b), the space \( X \) is normed, \( A \) is column-positive, and \( B \) satisfies (9), then \( g_{\infty, A, B}^{\Phi, p} \) is an F-norm. Moreover, \( g_{\infty, A, D(c_k)}^{\Phi, p} \) is an absolutely monotone F-seminorm (F-norm) on \( c_0[A, D(c_k), \Phi, p, X] \).

We may consider the space \( \ell'[A, B, \Phi, p, X] \) as the space \( \ell'[A^{1/r}, B, \Phi, p, X] \). So, since \( \ell' \) is solid AK-space with respect to the norm \( \|u\|_r = (\sum_k |u_k|^r)^{1/r} \), Proposition 1 b) gives the following corollary.

**Corollary 4.** Let \( \Phi, p, A, X, X, \) and \( B \) be the same as in Proposition 1.

If the matrix \( A \) is such that

\[
\sum_n |a_{nk}| < \infty \quad (k \in \mathbb{N}),
\]
then
\[ g_{1,A,B}^{\Phi,p} = \left( \sum_{n} |A^{1/r} (\Phi^p (Bx))|^r \right)^{1/r} \]
\[ = \left( \sum_{n} \sum_{k} a_{nk} \left( \phi_k \left( \left| \sum_{j} b_{kj} x_j \right| \right) \right)^{p_k} \right)^{1/r} \]
is an \( F \)-seminorm on \( \ell[A,B,\Phi,p,X] \) for an arbitrary sequence of moduli \( \Phi \).

If the space \( X \) is normed, \( A \) is column-positive, and (9) holds, then \( g_{1,A,B}^{\Phi,p} \) is an \( F \)-norm. Moreover, \( g_{1,A,D(c_k)}^{\Phi,p} \) is an absolutely monotone \( F \)-seminorm (or \( F \)-norm) on \( \ell[A,D(c_k),\Phi,p,X] \).

The GS spaces from Corollaries 3 and 4 have been studied earlier in the special cases when the role of matrix \( A \) was played not only by \( C_1 \) and different diagonal matrices, but also by the matrix of de la Vallée-Poussin and by the matrix of lacunary strong convergence. Recall that if \( d = (d_k) \) is a non-decreasing sequence of positive numbers tending to \( \infty \) with \( d_1 = 1 \) and \( d_{n+1} \leq d_n + 1 \), then the matrix of de la Vallée-Poussin \( V_d = (v_{nk}) \) is defined by the equalities \( v_{nk} = 1/d_n \) if \( k \in [n - d_n + 1, n] \) and \( v_{nk} = 0 \) otherwise. Further, a sequence of integers \( \theta = (k_j) \) is called lacunary if \( k_0 = 0, 0 < k_j < k_{j+1} \) and \( h_j = k_j - k_{j-1} \to \infty \) as \( j \to \infty \). A sequence \( u = (u_k) \) is said to be lacunary strongly convergent to a number \( l \) if (see [26])
\[ \lim_j 1/h_j \sum_{i \in (k_{j-1},k_j]} |u_i - l| = 0. \]

Thus, given the matrix \( N_\theta = (w_{ji}) \) with \( w_{ji} = 1/h_j \) if \( i \in (k_j-1,k_j] \) and by \( w_{ji} = 0 \) otherwise, the lacunary strong convergence is precisely the strong \( N_\theta \)-summability. It is clear that both matrices \( V_d \) and \( N_\theta \) are regular and column-positive. Moreover, \( V_d \) is normal and reduces to \( C_1 \) for \( d_n = n \).

Corollary 3 permits to define, for example, an \( F \)-seminorm on the sequence spaces \( c_0[V_d,B,\phi,p] \) and \( \ell_\infty[V_d,B,\phi,p] \) from [16], and an \( F \)-norm on the GS space \( c_0[N_\theta,I,\Phi,X] \) which is considered in [41] for a Banach space \( X \). Corollary 3 also contains, as special cases, the results about the topologization of some sequence spaces of type \( c_0[A,I,\phi,p] \) from [10], [14], and [39].

In Theorem 1 of [14] it was asserted that for any non-negative regular matrix \( A \) the space \( \ell_\infty[A,I,\phi;p] \) may be topologized by the paranorm
\[ g_{\infty,A}^{\phi,p}(u) = \sup_n \left( \sum_{k} a_{nk} (\phi(|u_k|))^{p_k} \right)^{1/r}. \]
if \( \inf_k p_k > 0 \). But it is possible to prove, as in Remark 2, that this is not true for a bounded modulus \( \phi \) if \( A = I \), and \( p_k = 1 \) (\( k \in \mathbb{N} \)). By
whenever $x$ completes Theorem 10 (ii) of [10]. We also remark that Corollary 3 b) gives, of conditions (M5 and (M6)). In the case $k = 1$, $x$ satisfies one of conditions (M5) and (M6). If $\inf x_k > 0$, then it suffices to assume that the modulus $\phi$, $\phi(x) = \phi_k, (k \in N)$ satisfies one of the sequence spaces $\ell_p, (k \in N)$, an $F$-semisubaddition on the space $\phi$, $\phi(x)$, from $[15]$. Corollary 4 generalizes the results from [6], [12], and [13], where the paranorm topologies are defined on $(A, B, \phi, X)$. We also remark that Corollary 3 b) gives, of conditions (M5) and (M6). If $\inf x_k > 0$, then it suffices to assume that the modulus $\phi$, $\phi(x) = \phi_k, (k \in N)$ satisfies one of conditions (M5) and (M6). If $\inf x_k > 0$, then it suffices to assume that the modulus $\phi$, $\phi(x) = \phi_k, (k \in N)$ satisfies one of conditions (M5) and (M6). If $\inf x_k > 0$, then it suffices to assume that the modulus $\phi$, $\phi(x) = \phi_k, (k \in N)$ satisfies one of conditions (M5) and (M6). If $\inf x_k > 0$, then it suffices to assume that the modulus $\phi$, $\phi(x) = \phi_k, (k \in N)$ satisfies one of conditions (M5) and (M6). If $\inf x_k > 0$, then it suffices to assume that the modulus $\phi$, $\phi(x) = \phi_k, (k \in N)$ satisfies one of conditions (M5) and (M6).
a) If the solid sequence space \( \lambda \) is topologized by an absolutely monotone \( F \)-seminorm \( g_\lambda \), then statements a) and b) of Proposition 1 hold with

\[
g_{\lambda,A,B}^\Phi (x) = \sum_{i=1}^{m} |x_i| + g_{\lambda,A,B}^\Phi (x)
\]

instead of \( g_{\lambda,A,B}^\Phi (x) \). Then \( g_{\lambda,A,B}^\Phi \) is an \( F \)-norm on \( \lambda[A,B,\Phi,p,X] \) whenever \( g_\lambda \) is \( F \)-norm, \( X \) is normed, and \( A \) is column-positive.

b) Statements of Corollaries 3 and 4 are true with

\[
g_{\nu,A,B}^\Phi (x) = \sum_{i=1}^{m} |x_i| + g_{\nu,A,B}^\Phi (x), \quad \nu \in \{ \infty, 1 \}
\]

instead of \( g_{\nu,A,B}^\Phi (x) \), \( \nu \in \{ \infty, 1 \} \). If \( X \) is normed and \( A \) is column-positive, then these functionals determine \( F \)-norms, respectively, on \( \ell_\infty[A,B,\Phi,p,X] \), \( c_0[A,B,\Phi,p,X] \), and \( \ell[A,B,\Phi,p,X] \).

In 1) and 2), the term \( \sum_{i=1}^{m} |x_i| \) may also be replaced with the expression \( \max_{i=1,...,m} |x_i| \) or, more generally, with the expressions \( \sum_{i=1}^{m} \varphi_i \left( |x_i| \right) \) or \( \max_{i=1,...,m} \varphi_i \left( |x_i| \right) \), where \( \varphi_i \) \( (i = 1, \ldots, m) \) are moduli.

For example, the authors of [1], [21], and [22] determine paranorms of type \( g_{\nu,A,B}^\Phi \) on some GS spaces \( c_0[A,B,\Phi,p,X] \) with \( B = v \Delta^m \). At the same time the various spaces of type \( c_0[A,v \Delta^m,\Phi,p,X] \) and \( \ell[A,v \Delta^m,\Phi,p,X] \) from [2], [3], [4], [8], [20], [24], [43], and [48] are topologized, as in Corollaries 3 and 4, by the paranorms \( g_{\nu,A,v \Delta^m}^\Phi (x) \) \( (\nu \in \{ \infty, 1 \}) \). Proposition 3 b) allows us to define alternative paranorms (or \( F \)-seminorms) in the form \( g_{\nu,A,v \Delta^m}^\Phi \) \( (\nu \in \{ 1, \infty \}) \) on all these spaces. In addition, Corollary 3 and Proposition 3 b) determine \( F \)-seminorm (or paranorm) topologies on the spaces of type \( \ell_\infty[A,v \Delta^m,\Phi,p,X] \) from the papers [1], [2], [4], [8], [20], [21], [22], [24], and [43].

Tripathy, Mahanta, and Et [50] consider the generalized sequence space \( m(\psi,p)[I,\Delta^n,\phi,X] \), where \( (m(\psi,p),g_{m(\psi,p)}) \) \( (1 \leq p < \infty) \) is the solid Banach space defined in [51] by means of a special non-decreasing sequence \( \psi = (\psi_k) \). Theorem 2 of [50] asserts that \( g_{m(\psi,p),I,\Delta^n}^\Phi \) is a paranorm on \( m(\psi,p)[I,\Delta^n,\phi,X] \) for any modulus \( \phi \). Besides this, Tripathy and Chandra ([48], Theorem 3.2) assert that the sequence space \( \ell_\infty[I,D(c_k)\Delta^n_1,\phi,p] \) may be topologized by the paranorm \( g_{\infty,1,D(c_k)\Delta^n_1}^\Phi \) for every modulus \( \phi \). But these assertions are not true in general. Indeed, if \( p = n = 1 \) and \( \psi_k = k \) \( (k \in \mathbb{N}) \), then (see [51], Corollary 11) \( m(\psi,p) = \ell_\infty \) with \( g_{m(\psi,p)} = || \cdot ||_\infty \). Hence \( m(\psi,p)[I,\Delta^n,\phi,K] \) reduces to the space

\[
\ell_\infty[I,\Delta^n,\phi] = \{ u = (u_k) \in s : \Delta^n u \in \ell_\infty(\phi) \},
\]
and

\[ \hat{g}_m^{\phi, I, \Delta_n}(u) = |u_1| + \sup_k |\Delta^1 u_k| \].

Analogously, for \( n = 1 \) and \( p_k = c_k = 1 \) (\( k \in \mathbb{N} \)), \( \ell_\infty[I, \Delta^1, \phi] \) reduces also to \( \ell_\infty[I, \Delta^1, \phi] \) with

\[ g_{\infty, I, D(c_k) \Delta^1_n}^{\phi, p} = \sup_k |\Delta^1 u_k| \].

Therefore, if the modulus \( \phi \) is bounded, then \( \ell_\infty[I, \Delta^1, \phi] = s \) and we can prove, as in Remark 2, that \( \hat{g}_m^{\phi, I, \Delta_n} \) and \( g_{\infty, I, D(c_k) \Delta^1_n}^{\phi, p} \) are not paranorms.

Proposition 3 a) and Corollary 3 a) show that Theorem 2 of [50], and Theorem 3.2 (about \( \ell_\infty[I, D(c_k) \Delta^1_n, \phi, p] \)) from [48], are true whenever the modulus \( \phi \) satisfies one of conditions (M5\textdegree) and (M6\textdegree).

Let us apply Proposition 2 and Corollary 2 to define F-seminorms and F-norms on the GS spaces

\[ U_{\ell_\infty}[A, B, \Phi, p, X] = \left\{ x \in s(X) : \sup_{n,i} \sum_k a_{nk} \left( \phi_k \left( \left| \sum_j b_{ij} x_j \right| \right) \right)^{p_k} < \infty \right\} \]

and

\[ U_{c_0}[A, B, \Phi, p, X] = \left\{ x \in s(X) : A(\Phi^p(Bx)) \in U_{c_0} \right\} = M_{c_0}[A, B, \Phi, p, X] \cap uc_0[A, B, \Phi, p, X], \]

where

\[ M_{c_0}[A, B, \Phi, p, X] = \left\{ x \in s(X) : A(\Phi^p(Bx)) \in M(X) \right\}, \]

\[ uc_0[A, B, \Phi, p, X] = \left\{ x \in s(X) : \lim_n \sum_k a_{nk} \left( \phi_k \left( \left| \sum_j b_{ij} x_j \right| \right) \right)^{p_k} = 0 \right\} \text{ uniformly in } i \] .

Taking into account the inclusion

\[ U_{c_0}[A, B, \Phi, p, X] \subset U_{\ell_\infty}[A, B, \Phi, p, X], \]

from Proposition 2 we get the following result.

**Corollary 5.** Let \( \Phi, p, A, B, X, \) and \( X \) be the same as in Proposition 2. a) If \( \Phi^{p/r} \) satisfies one of conditions (M5) and (M6), then on the GS space \( U_{\ell_\infty}[A, B, \Phi, p, X] \) we may define the F-seminorm

\[ g_{\infty, A, B}^{\Phi, p}(x) = \sup_{n,i} \sum_k a_{nk} \left( \phi_k \left( \left| \sum_j b_{ij} x_j \right| \right) \right)^{p_k}. \]
b) If the moduli \( \phi_k \) \( (k \in \mathbb{N}) \) are unbounded, and the row-finite and column-positive matrix \( A \) satisfies (14), then \( g^\phi_{\infty,A,B} \) is an F-seminorm on the space \( Uc_0[A,B,\Phi,\Phi,p,X] \).

If, in a) and b), \( X \) is normed and \( B \) satisfies (4), then \( g^\phi_{\infty,A,B} \) is an F-norm. Moreover, \( g^\phi_{\infty,A,B} \) is an absolutely monotone F-seminorm (or F-norm) on \( Uc_0[A,B,\Phi,\Phi,p,X] \) if \( B \) is a sequence of diagonal matrices.

Corollary 5 permits to determine F-seminorms (or paranorms), for example, on the spaces \( U\ell_\infty[V,X,F_{\ell_1},\Phi,X] \) and \( Uc_0[V,X,F_{\ell_1},\Phi,X] \) from [28], and also on similar spaces from [30].

For an infinite matrix \( B = (b_{nk}) \) let \( \hat{B} \) be the sequence of matrices \( \hat{B}^i = (b_{n+i,k})_{n,k \in \mathbb{N}} \) \( (i \in \mathbb{N}) \). In this case we have \( \hat{B}x = (B_{n+i}x)_{n,i \in \mathbb{N}} \), which, for \( B = I \), gives \( \hat{x} = (x_{n+i})_{n,i \in \mathbb{N}} \). We can prove a stronger variant of Corollary 5 b) under the assumption that \( p_k = 1, \phi_k = \phi \ (k \in \mathbb{N}), A = C_1 \) and \( B = \hat{B} \). Then the GS space \( uc_0[A,B,\Phi,\Phi,p,X] \) reduces to (for the case \( B = I \) and \( X = \mathbb{K} \) see [40])

\[
uc_0[C_1,\hat{B},\phi,X] = \left\{ x \in s(X) : \lim_n n^{-1} \sum_{k=1}^{n} \phi \left( |B_{k+i-1}x| \right) = 0 \right\}.
\]

uniformly in \( i \).

**Proposition 4.** Let \( B = (b_{nk}) \) be an \( m \)-normal infinite matrix such that \( K = \inf_n |b_{nn}| > 0 \) and there exists an index \( j_0 > m \) with \( b_{nk} = 0 \) \( (k \leq n + m - j_0, n > j_0 - m) \). The functional

\[
g^\phi_{\infty,C_1,\hat{B}}(x) = \sup_{n,i} n^{-1} \sum_{k=1}^{n} \phi \left( |B_{k+i-1}x| \right) = \sup_k \phi \left( |B_kx| \right)
\]

defines an F-seminorm (F-norm if \( X \) is normed and \( m = 0 \)) on the GS space \( uc_0[C_1,\hat{B},\phi,X] \) if and only if the modulus \( \phi \) is unbounded.

**Proof.** Since \( x \in uc_0[C_1,\hat{B},\phi,X] \) means that the sequence \( \phi(B_kx) = \left( \phi \left( |B_kx| \right) \right) \) is almost convergent to zero, but every almost convergent sequence is bounded (see [17], Theorem 1.2.18), we clearly have

\[
uc_0[C_1,\hat{B},\phi,X] = Uc_0[C_1,\hat{B},\phi,X].
\]

(16)

Moreover, by

\[
\phi \left( |B_i x| \right) \leq \sup_n n^{-1} \sum_{k=1}^{n} \phi \left( |B_{k+i-1}x| \right) \leq \sup_k \phi \left( |B_kx| \right) \quad (i \in \mathbb{N}),
\]

we get

\[
g^\phi_{\infty,C_1,\hat{B}}(x) = \sup_k \phi \left( |B_kx| \right).
\]

(17)
Sufficiency. If the modulus $\phi$ is unbounded, then $g_{\infty,C_1,\hat{B}}^\phi$ is an $F$-seminorm on $uc_0[C_1, \hat{B}, \phi, X]$ by Proposition 3 b) because the matrix $C_1$ is normal and regular. In particular, since 0-normal matrix is normal and every normal matrix $B$ satisfies (9), the functional $g_{\infty,C_1,\hat{B}}^\phi$ defines an $F$-norm on $uc_0[C_1, \hat{B}, \phi, X]$ if $X$ is normed and $m = 0$.

Necessity. Assume that $g_{\infty,C_1,\hat{B}}^\phi$ is an $F$-seminorm on $uc_0[C_1, \hat{B}, \phi, X]$ and define (see [35])

$$\hat{w}_0 = \{u = (u_k) \in s : \lim_{n} n^{-1} \sum_{k=1}^{n} |u_{k+i-1}| = 0 \text{ uniformly in } i\}.$$

If $v = (v_k)$ is the sequence of numbers

$$v_k = \begin{cases} 1, & \text{if } k = 2^j \ (j \in \mathbb{N}) \\ 0 & \text{otherwise}, \end{cases}$$

then for $2^j \leq n < 2^{j+1}$ we have

$$\sup_{i} n^{-1} \sum_{k=1}^{n} |v_{k+i-1}| \leq n^{-1} \sum_{k=1}^{n} |v_{k+1}| < \frac{j+1}{2^j} \to 0 \text{ as } j \to \infty,$$

and so, $v \in \hat{w}_0$. By means of $v$, using a fixed element $y_0 \in X$ with $\|y_0\| = 1$, we consider the $X$-valued sequence $y = (y_k)$, $y_k = kv_ky_0 \ (k \in \mathbb{N})$. Now, assuming that the modulus $\phi$ is bounded and $M = \sup_{t>0} \phi(t)$, by the inequalities

$$\phi \left( |y_k| \right) \leq \phi(k) \leq M v_k \ (k \in \mathbb{N})$$

we get $\phi(y) \in \hat{w}_0$ because $\hat{w}_0$ is solid sequence space. Further, the equality $Bz = y$ clearly determines a new $X$-valued sequence $z = (z_k)$ with $z_1 = \cdots = z_m = 0$. This sequence $z$ is unbounded, since we can find an index $i_0$ such that, for $i > i_0$,

$$B_{2^i}z = b_{2^i,2^i}z_{2^i} = y_{2^i} = 2^i y_0,$$

and so, $|z_{2^i}| = 2^i |b_{2^i,2^i}|^{-1} \geq 2^i / K$ if $i > i_0$. Moreover, $z \in uc_0[C_1, \hat{B}, \phi, X]$ by the representation

$$uc_0[C_1, \hat{B}, \phi, X] = \hat{w}_0(B, \phi, X).$$

Thus, as in Remark 2, using equality (17) we can show, that $g_{\infty,C_1,\hat{B}}^\phi$ does not satisfy axiom (N4), i.e., it is not an $F$-seminorm. $\square$

Assumptions of Proposition 4 are clearly satisfied for $B = I$ and $B = \Delta^m$. 

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Corollary 6. The functional
\[ g^\phi_{\infty,C_1,\hat{I}}(x) = \sup_n n^{-1} \sum_{k=1}^n \phi \left( |x_{k+i-1}| \right) = \sup_k \phi \left( |x_k| \right) \]
defines an F-seminorm (F-norm if \( X \) is normed) on the GS space
\[ uc_0[C_1,\hat{I},\phi,X] = \{ x \in s(X) : \lim_n n^{-1} \sum_{k=1}^n \phi \left( |x_{k+i-1}| \right) = 0 \text{ uniformly in } i \} \]
if and only if the modulus \( \phi \) is unbounded.

Et ([23], Theorem 2.3) asserts that the sequence space
\[ \hat{c},\phi,p](\Delta^m) = \left\{ u \in s : \lim_n 1/n \sum_{k=1}^n \phi(\Delta^m u_{k+i})^{p_k} = 0 \text{ uniformly in } i \right\} \]
may be topologized by the paranorm
\[ g_\Delta(u) = \sup_{n,i} \left( \sum_{k=1}^n \phi(\Delta^m u_{k+i})^{p_k} \right)^{1/r} \]
for any modulus \( \phi \). Proposition 4 (with \( B = \Delta^m \)) shows that this is not true if \( \phi \) is bounded, since the space \([\hat{c},\phi,p](\Delta^m)\) reduces, for \( p_k = 1 \ (k \in \mathbb{N}) \), to \( uc_0[C_1,\Delta^m,\phi,\mathbb{K}] \) with \( g_\Delta = g^\phi_{\infty,C_1,\Delta^m} \). Corollary 6 allows us to say that similar inaccuracies may be found in theorems about the topologization of various spaces of type \( uc_0[A,B,\Phi,p,X] \) from [7], [19], [36], [38], and [45], because all these spaces contain \( uc_0[C_1,\hat{I},\phi,X] \) as a special case.

Remark 3. For the topologization of GS spaces \( uc_0[A,B,\Phi,p,X] \) by F-seminorms (or paranorms) \( g^{\Phi,p}_{\infty,A,B} \) it is necessary that
\[ uc_0[A,B,\Phi,p,X] \subset U_{\ell_\infty}[A,B,\Phi,p,X] \] (18)
or, equivalently,
\[ uc_0[A,B,\Phi,p,X] = U_{c_0}[A,B,\Phi,p,X]. \]
The following example shows that (18) is not true in general. Let \( A_1 = (a_{nk}) \) be the Cesàro matrix \( C_1 = (c_{nk}) \) which is modified by setting \( c_{1k} = 0 \ (k \geq 2) \), and let \( \Phi = (\phi_1, \phi, \phi, \ldots) \) be the sequence of moduli, where \( \phi_1(t) = t \) and \( \phi \) is a bounded modulus. Then the unbounded sequence \( y \), defined in the proof of Proposition 4, belongs to \( uc_0[A_1,\hat{I},\Phi,X] \) because, for \( n \geq 2 \), we have
\[ \sum_k a_{nk}\phi_k \left( |y_{k+i-1}| \right) = n^{-1} \sum_{k=2}^n \phi \left( |y_{k+i-1}| \right). \]
But since (for $n = 1$)
\[
\sup_i \sum_k a_{1k} \phi_k \left( |y_{k+i-1}| \right) = \sup_i |y_i| = \infty,
\]
y is not in $U_{\ell_\infty}[A_1, \hat{I}, \Phi, X]$. This example allows us to state that the proofs of inclusions (18) from [19], [38], and [45] are not convincing, and the correctness of the definition of functional $g^{\Phi,P}_{\infty,A,B}$ in [7] remains actually open.

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**References**


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